Math 1080: Spring 2011

Homework #11 (due April 22)

Problem 1:

a) Determine the matrices **U**, $\mathbf{\Sigma}$, **V** in the singular value decomposition, $\mathbf{A} = \mathbf{U\Sigma V}^{\mathrm{T}}$, of the following matrices

$$\mathbf{A}_1 = \begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} -2 & 2 & 0 & -2 \\ 2 & -1 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ -2 & 3 & 1 & -1 \end{bmatrix}$$

b) Use the results of a) to find, in each case, the rank of $\mathbf{A}_i$ the norm $\|\mathbf{A}_i\|$ and the basis for null($\mathbf{A}_i$).

SOLUTION:

a)

<u>Case 1:</u>

$$\mathbf{A}_1\mathbf{A}_1^{\mathrm{T}} = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 4 & -1 \\ -1 & -1 & 3 \end{bmatrix},$$

$$\det(\mathbf{A}_1\mathbf{A}_1^{\mathrm{T}} - \lambda\mathbf{I}) = \begin{vmatrix} 4-\lambda & 0 & -1 \\ 0 & 4-\lambda & -1 \\ -1 & -1 & 3-\lambda \end{vmatrix} = (4-\lambda)^2(3-\lambda) - 2(4-\lambda)$$

$$= (4-\lambda)(\lambda^2 - 7\lambda + 10) = (4-\lambda)(\lambda-2)(\lambda-5)$$

The eigenvalues of $\mathbf{A}_1\mathbf{A}_1^{\mathrm{T}}$ are $\lambda_i = 5$, 4, and 2 and hence the singular values of $\mathbf{A}_1$ are $\sqrt{5}, 2,$ and $\sqrt{2}$ and

$$\mathbf{\Sigma} = \begin{bmatrix} \sqrt{5} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \end{bmatrix}$$

For $\lambda_1 = 5$ the eigenvector of $\mathbf{A}_1\mathbf{A}_1^{\mathrm{T}}$ obeys $\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ -1 & -1 & -2 \end{bmatrix}\mathbf{x}_1 = \mathbf{0}$, i.e., $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

For $\lambda_2 = 4$ the eigenvector of $\mathbf{A}_1\mathbf{A}_1^{\mathrm{T}}$ obeys $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -1 & -1 & -1 \end{bmatrix}\mathbf{x}_2 = \mathbf{0}$, i.e., $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

For $\lambda_3 = 2$ the eigenvector of $\mathbf{A}_1\mathbf{A}_1^{\mathrm{T}}$ obeys $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}\mathbf{x}_3 = \mathbf{0}$, i.e., $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

The columns of **U** are the normalized eigenvectors $\mathbf{x}_i$: $\mathbf{U} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$

$$\mathbf{A}_1\mathbf{A}_1^{\mathrm{T}} = \begin{bmatrix} 3 & 1 & -1 & 0 \\ 1 & 3 & 1 & 0 \\ -1 & 1 & 3 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

The eigenvalues of $\mathbf{A}_1\mathbf{A}_1^{\mathrm{T}}$ are $\lambda_i = 5, 4, 2,$ and 0

For $\lambda_1 = 5$ the eigenvector obeys $\begin{bmatrix} -2 & 1 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 1 & -2 & 2 \\ 0 & 0 & 2 & -3 \end{bmatrix}\mathbf{x}_1 = \mathbf{0}$, i.e., $\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 3 \\ 2 \end{bmatrix}$

For $\lambda_2 = 4$ the eigenvector obeys $\begin{bmatrix} -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 2 \\ 0 & 0 & 2 & -2 \end{bmatrix}\mathbf{x}_2 = \mathbf{0}$, i.e., $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

For $\lambda_3 = 2$ the eigenvector obeys $\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & 1 & 1 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}\mathbf{x}_3 = \mathbf{0}$, i.e., $\mathbf{x}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

For $\lambda_4 = 0$ the eigenvector obeys $\begin{bmatrix} 3 & 1 & -1 & 0 \\ 1 & 3 & 1 & 0 \\ -1 & 1 & 3 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix} \mathbf{x}_4 = \mathbf{0}$, i.e., $\mathbf{x}_4 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}$

The columns of $\mathbf{V}$ are the normalized eigenvectors $\mathbf{x}_i$ multiplied by ± 1 so that $\mathbf{A}_1 \mathbf{v}_i = \sigma_i \mathbf{u}_i$:

$$\mathbf{V} = \begin{bmatrix} \frac{1}{\sqrt{15}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{10}} \\ \frac{-1}{\sqrt{15}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{10}} \\ \frac{-3}{\sqrt{15}} & 0 & 0 & \frac{2}{\sqrt{10}} \\ \frac{-2}{\sqrt{15}} & 0 & \frac{-1}{\sqrt{3}} & \frac{-2}{\sqrt{10}} \end{bmatrix}$$

<u>Case 2:</u>

The matrix $\mathbf{A}_2$ is symmetric and hence it has the orthogonal decomposition $\mathbf{A} = \mathbf{Q\Lambda Q}^{\mathrm{T}}$ which yields immediately a singular value decomposition with $\mathbf{U} = \mathbf{Q}$, $\mathbf{\Sigma} = |\mathbf{\Lambda}|$, and $\mathbf{V} = \mathbf{Q}\,\mathrm{sign}(\mathbf{\Lambda})$.

$$\det(\mathbf{A}_2 \mathbf{A}_2^{\mathrm{T}} - \lambda \mathbf{I}) = \begin{vmatrix} -2-\lambda & 2 & 0 & -2 \\ 2 & -1-\lambda & 1 & 3 \\ 0 & 1 & 1-\lambda & 1 \\ -2 & 3 & 1 & -1-\lambda \end{vmatrix}$$

$$= -(2+\lambda) \begin{vmatrix} -1-\lambda & 1 & 3 \\ 1 & 1-\lambda & 1 \\ 3 & 1 & -1-\lambda \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 & 3 \\ 0 & 1-\lambda & 1 \\ -2 & 1 & -1-\lambda \end{vmatrix}$$

$$+ 2 \begin{vmatrix} 2 & -1-\lambda & 1 \\ 0 & 1 & 1-\lambda \\ -2 & 3 & 1 \end{vmatrix}$$

$$= \lambda^2 (\lambda^2 + 3\lambda - 18)$$

$$= \lambda^2 (\lambda - 3)(\lambda + 6)$$

The eigenvalues are -6, 3, 0 and 0, and hence $\mathbf{\Sigma} = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

For $\lambda_1 = -6$ the eigenvector obeys $\begin{bmatrix} 4 & 2 & 0 & -2 \\ 2 & 5 & 1 & 3 \\ 0 & 1 & 7 & 1 \\ -2 & 3 & 1 & 5 \end{bmatrix} \mathbf{x}_1 = \mathbf{0}$, i.e., $\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

For $\lambda_2 = 3$ the eigenvector obeys $\begin{bmatrix} -5 & 2 & 0 & -2 \\ 2 & -4 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ -2 & 3 & 1 & -4 \end{bmatrix} \mathbf{x}_2 = \mathbf{0}$, i.e., $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda_3 = 0$ the eigenvector obeys $\begin{bmatrix} -2 & 2 & 0 & -2 \\ 2 & -1 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ -2 & 3 & 1 & -1 \end{bmatrix} \mathbf{x}_1 = \mathbf{0}$, i.e., $\mathbf{x}_3 = \begin{bmatrix} -s+t \\ t \\ -s-t \\ s \end{bmatrix}$.

We pick, for example $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{x}_4 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

Thus, $\mathbf{U}_2 = \mathbf{Q} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ -1 & 1 & 0 & 1 \end{bmatrix}$, $\mathbf{V}_2 = \mathbf{Q}\,\text{sign}(\mathbf{\Lambda}) = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 1 & -1 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

b)

$\text{rank}(\mathbf{A}_1) =$ # of nonzero singular values = 3, $\|\mathbf{A}_1\| = \sigma_1 = \sqrt{5}$,

$\text{null}(\mathbf{A}_1) = \left\langle \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix} \right\rangle$

$\text{rank}(\mathbf{A}_2) = 2$, $\|\mathbf{A}_1\| = \sigma_1 = 6$, $\text{null}(\mathbf{A}_1) = \left\langle \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\rangle$

Computer Assignment 7:

a) Write a MATLAB function **`[U,S,V]=svdsimp(A)`** that computes the singular value decomposition, $\mathbf{A} = \mathbf{USV}^T$, of an $m \times n$ matrix **A**. The output variables are the $m \times m$ orthogonal matrix **U**, the $m \times n$ diagonal matrix **S**, and the $n \times n$ orthogonal matrix **V**.

b) Use the function **`svdsimp`** to calculate the signular values of the following matrix

$$\mathbf{A} = \begin{bmatrix} -3 & 2 & 6 & -8 & 0 & 5 & 3 \\ 2 & -9 & 5 & 3 & 4 & 4 & 1 \\ 0 & 0 & 4 & 2 & -1 & -1 & -1 \\ 3 & 2 & 0 & -3 & -3 & 1 & 1 \\ 5 & 4 & 4 & 4 & 5 & 8 & 9 \end{bmatrix}$$

PROGRAM:

```
function [U,S,V] = svdsimp(A)
[m,n] = size(A);
[V,D] = eig(A'*A);
[d,ind] = sort(diag(D),'descend');
V = V(:,ind);
[U,S] = qr(A*V);
end
```

OUTPUT:

```
 >> A = [3 -2 6 -8 0 5 3; 2 -9 5 3 4 4 1; 0 0 4 2 -1 -1 -1; 3 2 0 -3 -3
1 1; 5 4 4 4 5 8 9];
>> [U,S,V] = svdsimp(A)

U =

    0.4430   -0.1948   -0.8077    0.0022    0.3367
    0.4446   -0.7730    0.2944   -0.1111   -0.3252
    0.0299   -0.1243    0.0783    0.9862    0.0701
    0.0484    0.2300   -0.3952    0.1215   -0.8797
    0.7764    0.5442    0.3139    0.0168    0.0463


S =

   17.8860   -0.0000    0.0000   -0.0000    0.0000   -0.0000    0.0000
         0   11.3687    0.0000    0.0000    0.0000    0.0000    0.0000
         0         0   10.9608   -0.0000    0.0000   -0.0000   -0.0000
         0         0         0    4.5280    0.0000    0.0000   -0.0000
         0         0         0         0    2.8638    0.0000   -0.0000
```

```
V =

    0.3492    0.1126   -0.1323    0.0515   -0.7151   -0.5075   -0.2761
   -0.0942    0.8782   -0.0519    0.2884    0.2373   -0.0098   -0.2788
    0.4532   -0.2950   -0.1647    0.7664    0.3002   -0.0550   -0.0017
    0.0453    0.0420    0.9072    0.2924   -0.2461    0.1610    0.0356
    0.3067   -0.0824    0.3517   -0.3778    0.5235   -0.4808   -0.3571
    0.5716    0.0565   -0.0751   -0.2569   -0.0688    0.6866   -0.3496
    0.4909    0.3426    0.0203   -0.1806    0.0529   -0.1041    0.7714


>> norm(U*S*V'-A)

ans =

  1.5274e-014
```