**Math 1080: Spring 2010**
**Homework #10 (due April 15)**

**Problem 1:**

Let **Q** and **R** be the QR factors of a symmetric tridiagonal matrix **H**. Show that the product $\mathbf{K} = \mathbf{RQ}$ is again a symmetric tridiagonal matrix.

(Hint: Prove the symmetry of **K**. Show that **Q** has Hessenberg form and that the product of an upper triangular matrix and a Hessenberg matrix is again a Hessenberg matrix. Then use the symmetry of **K**.)

*SOLUTION:*

Symmetry of **K**:

From the symmetry of **H** it follows that $\mathbf{QR} = \mathbf{H} = \mathbf{H}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}$. Using this relation we find that

$$\mathbf{K} = \mathbf{RQ} = \mathbf{Q}^{\mathrm{T}}\mathbf{QRQ} = \mathbf{Q}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{Q}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} = \mathbf{K}^{\mathrm{T}}$$

and hence **K** is symmetric.

Hessenberg form of **Q**:

The $m \times m$ matrix **H** is tridiagonal and hence it has a Hessenberg form which implies that $h_{ij} = 0$ whenever $i > j+1$. The matrix **R** of the QR factorization of **H** is upper triangular end hence $r_{ij} = 0$ whenever $i > j$. Suppose $i > j+1$. We have

$$h_{ij} = \sum_{k=1}^{m} q_{ik} r_{kj} = q_{i1} r_{1j} + q_{i2} r_{2j} + \ldots + q_{ij} r_{jj}$$

When $j = 1$ the relation above has only one term on the right hand side, i.e, $h_{i1} = q_{i1} r_{11}$. Because $r_{11} \neq 0$ we can conclude that $q_{i1} = 0$ whenever $h_{i1} = 0$, i.e., whenever $i > 2$. When $j = 2$ we have $h_{i2} = q_{i1} r_{12} + q_{i2} r_{22}$ which, for $i > 2$, reduces to $h_{i2} = q_{i2} r_{22}$. Thus, again, $q_{i2} = 0$ whenever $h_{i2} = 0$, i.e., whenever $i > 3$. By continuing this argument we see that the matrix **Q** has zeros below the first subdiagonal just as **H** does and hence **Q** has Hessenberg form.

Hessenberg form of **K**:

Similarly as above, the matrix **R** is upper triangular with $r_{ij} = 0$ whenever $i > j$ and **Q** is Hessenberg with $q_{ij} = 0$ whenever $i > j+1$ and hence

$$k_{ij} = \sum_{k=1}^{m} r_{ik} q_{kj} = \sum_{k=i}^{j+1} r_{ik} q_{kj}$$

It follows that when $i > j+1$ the sum in the relation above has no terms and hence $k_{ij} = 0$. Thus, **K** is a Hessenberg matrix.

Because **K** is both Hessenberg and symmetric, it follows that **K** is tridiagonal.

**Problem 2:**
Reduce the following matrix to Hessenberg form.

$$\mathbf{A} = \begin{bmatrix} 3 & 7 & -2 & -8 \\ -2 & 1 & 1 & 4 \\ 4 & 9 & 0 & 0 \\ 4 & -4 & -4 & 2 \end{bmatrix}$$

*SOLUTION:*

Step 1:

$$\mathbf{b}_1 = \begin{bmatrix} -2 \\ 4 \\ 4 \end{bmatrix}, \quad \mathbf{v} = \|\mathbf{b}_1\|\mathbf{e}_1 - \mathbf{b}_1 = \begin{bmatrix} -4 \\ -4 \\ -4 \end{bmatrix}, \quad \mathbf{v}^T\mathbf{v} = 48$$

$$F = I - 2\mathbf{v}\mathbf{v}^T/(\mathbf{v}^T\mathbf{v}) = \frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/3 & -2/3 & -2/3 \\ 0 & -2/3 & 1/3 & -2/3 \\ 0 & -2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$H = Q_1\mathbf{A}Q_1^T = \begin{bmatrix} 3 & 9 & 0 & -6 \\ -6 & -3 & 3 & 0 \\ 0 & 3 & 0 & -6 \\ 0 & 0 & 6 & 6 \end{bmatrix}$$

Hessenberg form is achieved after one step.

**Computer Assignment 7:**

a) Write a MATLAB function **`[Q,L]=qralgshift(A,eps)`** that computes the eigenvalues and eigenvectors of a square, symmetric $m \times m$ matrix **A** using QR algorithm with shift. The output variables are the $m \times m$ orthogonal matrix **Q** which columns are the eigenvectors of **A** and $m \times m$ matrix **L** that has the corresponding eigenvalues of **A** on the main diagonal. The program should terminate iteration when the norm of the offdiagonal elements of $\mathbf{A}^{(k)}$, i.e., `norm(A-diag(diag(A))),` is smaller than $10^{-6}$.

b) Write a MATLAB function **`[Q,H]=hessenberg(A)`** that computes the Hessenberg form of $m \times m$ matrix **A**. The output variables are $m \times m$ orthogonal matrix **Q** and $m \times m$ matrix **H** similar to **A** that is in Hessenberg form.

c) Calculate the eigenvalues and eigenvectors of using (1) **`qralgshift`** algorithm only and (2) a combination of **`hessenberg`** and **`qralgshift`**.

$$\mathbf{A} = \begin{bmatrix} 3 & -2 & 12 & -9 & -6 & -9 \\ -2 & 3 & 1 & 8 & -1 & -1 \\ 12 & 1 & -9 & -3 & -3 & -11 \\ -9 & 8 & -3 & 4 & 4 & 17 \\ -6 & -1 & -3 & 4 & 15 & -3 \\ -9 & -1 & -11 & 17 & -3 & 7 \end{bmatrix}$$

Record the number of iterations needed to achieve the desired accuracy. Compare the two methods.

PROGRAMS:

```
function [Q,L]=qralgshift(A,eps)
n = length(A);
Qt = eye(n);
k = 0;
while norm(A - diag(diag(A))) > eps
    mu = A(n,n);
    [Q,R] = qr(A-mu*eye(n))
    A = R*Q+mu*eye(n)
    Qt = Qt*Q;
    k = k+1
    plot(ones(1,n)*k,diag(A),'.')
end
Q = Qt;
L = A;
disp(k)

function [Q,H] = hessenberg(A)
n = length(A(:,1));
Q = eye(n);
for k = 1:n-2
    x = A(k+1:n,k);
    v = x;
    v(1) = sign(x(1))*norm(x) + v(1);
    v = v/norm(v);
    A(k+1:n,k:n) = A(k+1:n,k:n) - 2*v*(v'*A(k+1:n,k:n));
    A(1:n,k+1:n) = A(1:n,k+1:n) - 2*(A(1:n,k+1:n)*v)*v';
    Q(:,k+1:n) = Q(:,k+1:n) - 2*(Q(:,k+1:n)*v)*v';
end
H = A;
```

OUTPUT:

```
>> A = [3 -2 12 -9 -6 -9; -2 3 1 8 -1 -1; 12 1 -9 -3 -3 -11; -9 8 -3 4 4 17;
-6 -1 -3 4 15 -3; -9 -1 -11 17 -3 7]

>> [Q,L]=qralgshift(A,1e-6)
   173

Q =

    0.4739    0.3194    0.4119    0.1278   -0.6821   -0.1488
   -0.1289   -0.1062    0.3821    0.0891    0.2947   -0.8552
    0.3181   -0.7126   -0.4578    0.0197   -0.3255   -0.2741
   -0.5375    0.4047   -0.5154    0.1057   -0.4039   -0.3277
   -0.2026   -0.0947    0.1736   -0.9264   -0.2409   -0.0597
   -0.5724   -0.4542    0.4230    0.3255   -0.3475    0.2458

L =

   35.2416    0.0000   -0.0000   -0.0000    0.0000    0.0000
    0.0000  -19.9354   -0.0000    0.0000    0.0000   -0.0000
    0.0000   -0.0000  -12.7052    0.0000   -0.0000    0.0000
    0.0000    0.0000   -0.0000   16.5855   -0.0000   -0.0000
   -0.0000   -0.0000    0.0000   -0.0000   -2.4427   -0.0000
         0         0         0         0         0    6.2562


>> [Q,H] = hessenberg(A)

Q =

    1.0000         0         0         0         0         0
         0   -0.1075   -0.0470   -0.2157    0.4055   -0.8805
         0    0.6451   -0.7340    0.0497   -0.1628   -0.1267
         0   -0.4838   -0.3443   -0.6024   -0.5329   -0.0204
         0   -0.3226   -0.0664    0.7320   -0.4781   -0.3566
         0   -0.4838   -0.5797    0.2286    0.5445    0.2848


H =

    3.0000   18.6011         0         0         0         0
   18.6011   19.2052   14.3889   -0.0000    0.0000   -0.0000
         0   14.3889   -6.1145    7.9925    0.0000    0.0000
         0         0    7.9925    2.9420  -13.2361    0.0000
         0         0   -0.0000  -13.2361   -2.0517    6.0186
         0         0   -0.0000         0    6.0186    6.0189

>> [Q,L]=qralgshift(H,1e-6)
   172


Q =

    0.4739   -0.3194   -0.4119   -0.1278   -0.6821    0.1488
    0.8215    0.3938    0.3477   -0.0934    0.1996    0.0260
    0.3029   -0.6583   -0.2388    0.1822    0.5815   -0.2157
    0.0882    0.4294   -0.4291    0.6857   -0.0922   -0.3808
   -0.0324    0.3447   -0.6515   -0.5967    0.3137   -0.0349
```

```
   -0.0067   -0.0799    0.2094   -0.3399   -0.2231   -0.8857


L =

   35.2416    0.0000   -0.0000   -0.0000    0.0000    0.0000
    0.0000  -19.9354    0.0000    0.0000    0.0000    0.0000
         0    0.0000  -12.7052    0.0000    0.0000   -0.0000
         0         0    0.0000   16.5855   -0.0000   -0.0000
         0         0   -0.0000   -0.0000   -2.4427    0.0000
         0         0         0         0         0    6.2562
```

Hessenberg form does not result in a significant speedup of the method. In fact, QR algorithm with shifts requires more iterations than the simple QR algorithm (105) or QR algorithm applied to the Hessenberg form (84).