Due Friday, April 27, by 12pm (noon). If I am not in the office, put the exam in a sealed envelope and leave it in my mailbox or slide under the door. Let me know by e-mail that you did so. Make a copy of the exam for you to keep before you hand it in.

Problem 1: Solve the initial-value problem and verify your solution:

\[ xu_x + yu_y + u_z = u, \quad u(x, y, 0) = h(x, y) \]

Problem 2: Let \( u \in C^1(R^2) \) be a solution of \( u_y + uu_x = 0 \) in each of two regions separated by a curve \( x = \xi(y) \). Let \( u \) be continuous, but \( u_x \) have a jump discontinuity on the curve. Show that

\[ \frac{d\xi}{dy} = u \]

and hence that the curve is a characteristic. [Begin by showing \( (u_y^+ - u_y^-) + (u_x^+ - u_x^-) = 0 \).]

Problem 3: Let \( L = \Delta + c \) in 3 dimensions where \( c > 0 \) is a constant.

(a) Find all solutions of \( Lu = 0 \) with spherical symmetry.

(b) Show that

\[ K(x, \xi) = -\frac{4\pi r}{\cos(\sqrt{cr})}, \quad r = |x - \xi| \]

is a fundamental solution for \( L \) with pole \( \xi \).

(c) Show that for a solution \( u \) of \( Lu = 0 \) the following formula holds:

\[ u(\xi) = - \int_{\partial \Omega} K(x, \xi) \frac{du(x)}{dn_x} - u(x) \frac{dK(x, \xi)}{dn_x} ds \]

(d) Show that a solution \( u \) of \( Lu = 0 \) in the ball \( |x - \xi| \leq \rho \) for \( \sin(\sqrt{c}\rho) \neq 0 \) has the modified mean value property

\[ u(\xi) = \frac{\sqrt{c}\rho}{\sin(\sqrt{c}\rho)} \frac{1}{4\pi\rho^2} \int_{|x - \xi| = \rho} u(x) ds \]

(e) Show that for \( c > 0 \) there are solutions vanishing on a sphere but not in the interior.
Math 2900, Spring 2008

Take-home Final Exam
Instructor: D. Swigon

**Problem 4:** Consider the equation of elastic waves in 3 dimensional space with positive constants \( c_1, c_2 \)

\[
Lu = \left( \frac{\partial^2}{\partial t^2} - c_1^2 \Delta \right) \left( \frac{\partial^2}{\partial t^2} - c_2^2 \Delta \right) u(x,t) = 0
\]
(#)

(a) Show that the spherical mean \( M_u(x,r,t) \) of \( u \) obeys

\[
\Lambda r M_u = 0, \quad \Lambda = \left( \frac{\partial^2}{\partial t^2} - c_1^2 \frac{\partial^2}{\partial r^2} \right) \left( \frac{\partial^2}{\partial t^2} - c_2^2 \frac{\partial^2}{\partial r^2} \right)
\]

(b) Show that the general solution \( v(r,t) \) of \( \Lambda v = 0 \) is of the form

\[
v = F_1(r + c_1 t) + F_2(r - c_1 t) + G_1(r + c_2 t) + G_2(r - c_2 t)
\]

(c) Solve the general initial value problem for (#) using (a) and (b).

**Problem 5:** Show that when Gårding’s condition is satisfied, the solution \( u \) of the initial value problem

\[
P(D, \tau)u = 0 \quad \text{for } t \geq 0
\]
\[
\tau^k u = 0 \quad \text{for } k = 0, \ldots, m - 2 \text{ and } t = 0
\]
\[
\tau^{m-1} u = g(x) \quad \text{for } t = 0
\]
can be written

\[
u(x,t) = (1 - \Delta_x)^s \int K(x-y,t)g(y)dy
\]

where

\[
K(x-y,t) = (2\pi)^{-n/2} \int e^{i(x-y)\xi} (1 + |\xi|^2)^{-s} Z(\xi,t)d\xi
\]

and \( s \) is any integer larger than \( n/2 \).

**Problem 6:** Show that for \( n = 1 \) the solution of the initial value problem

\[
u_t - \Delta u = 0 \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}
\]
\[
u(x,0) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}
\]
is given by

\[
u(x,t) = \frac{1}{2} \left[ 1 + \phi \left( \frac{x}{\sqrt{4t}} \right) \right]
\]

where \( \phi(s) \) is the error function:

\[
\phi(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-t^2} dt
\]