Math 2900, Spring 2008

Take-home Final Exam

Instructor: D. Swigon

Due Friday, April 27, by 12pm (noon). If I am not in the office, put the exam in a sealed envelope and leave it in my mailbox or slide under the door. Let me know by e-mail that you did so. Make a copy of the exam for you to keep before you hand it in.

Problem 1: Solve the initial-value problem and verify your solution:

$$xu_x + yu_y + u_z = u, \qquad u(x,y,0) = h(x,y)$$

Problem 2: Let $u \in C^1(R^2)$ be a solution of $u_y + uu_x = 0$ in each of two regions separated by a curve $x = \xi(y)$. Let u be continuous, but u_x have a jump discontinuity on the curve. Show that

$$\frac{d\xi}{dy} = u$$

and hence that the curve is a characteristic.[Begin by showing $(u_y^+ - u_y^-) + u\,(u_x^+ - u_x^-) = 0$.]

Problem 3: Let $L = \Delta + c$ in 3 dimensions where $c > 0$ is a constant.

(a) Find all solutions of $Lu = 0$ with spherical symmetry.

(b) Show that

$$K(x,\xi) = -(4\pi\, r)^{-1}\cos(\sqrt{c}r), \qquad r = |x - \xi|$$

is a fundamental solution for L with pole ξ.

(c) Show that for a solution u of $Lu = 0$ the following formula holds:

$$u(\xi) = -\int_{\partial\Omega}\left(K(x,\xi)\frac{du(x)}{dn_x} - u(x)\frac{dK(x,\xi)}{dn_x}\right)dS_x$$

(d) Show that a solution u of $Lu = 0$ in the ball $|x - \xi| \le \rho$ for $\sin(\sqrt{c}\rho) \ne 0$ has the modified mean value property

$$u(\xi) = \frac{\sqrt{c}\rho}{\sin(\sqrt{c}\rho)}\frac{1}{4\pi\rho^2}\int_{|x-\xi|=\rho} u(x)dS_x$$

(e) Show that for $c > 0$ there are solutions vanishing on a sphere but not in the interior.

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Problem 4: Consider the equation of elastic waves in 3 dimensional space with positive constants c_1, c_2

$$Lu = \left(\frac{\partial^2}{\partial t^2} - c_1^2 \Delta\right)\left(\frac{\partial^2}{\partial t^2} - c_2^2 \Delta\right) u(x,t) = 0 \qquad (\#)$$

(a) Show that the spherical mean $M_u(x,r,t)$ of u obeys

$$\Lambda r M_u = 0, \qquad \Lambda = \left(\frac{\partial^2}{\partial t^2} - c_1^2 \frac{\partial^2}{\partial r^2}\right)\left(\frac{\partial^2}{\partial t^2} - c_2^2 \frac{\partial^2}{\partial r^2}\right)$$

(b) Show that the general solution $v(r,t)$ of $\Lambda v = 0$ is of the form

$$v = F_1(r + c_1 t) + F_2(r - c_1 t) + G_1(r + c_2 t) + G_2(r - c_2 t)$$

(c) Solve the general initial value problem for (#) using (a) and (b).

Problem 5: Show that when Gårding's condition is satisfied, the solution u of the initial value problem

$$\begin{aligned} &P(D,\tau)u = 0 && \text{for } t \ge 0 \\ &\tau^k u = 0 && \text{for } k = 0,\dots,m-2 \text{ and } t = 0 \\ &\tau^{m-1} u = g(x) && \text{for } t = 0 \end{aligned}$$

can be written

$$u(x,t) = (1 - \Delta_x)^s \int K(x-y,t) g(y) dy$$

where

$$K(x-y,t) = (2\pi)^{-n/2} \int e^{i(x-y)\cdot\xi} (1 + |\xi|^2)^{-s} Z(\xi,t) d\xi$$

and s is any integer larger than $n/2$.

Problem 6: Show that for $n = 1$ the solution of the initial value problem

$$u_t - \Delta u = 0 \quad \text{for } t > 0 \text{ and } x \in R$$

$$u(x,0) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

is given by

$$u(x,t) = \frac{1}{2}\left[1 + \phi\left(\frac{x}{\sqrt{4t}}\right)\right]$$

where $\phi(s)$ is the error function:

$$\phi(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-t^2} dt$$