

1 Line integrals

In Calc III you probably encountered what are called “line integrals”. First I will quickly review this. We had the following definition earlier.

Definition 1 *A smooth curve in R^n is a function $\gamma : [a, b] \rightarrow R^n$, where $[a, b]$ is some compact interval of R^1 , γ is continuously differentiable on $[a, b]$, and $|\gamma'(t)| \neq 0$ for $t \in [a, b]$.*

Note the distinction between the curve, which is a function, and the image of the function, $\gamma([a, b])$, which is a set of points in R^n . Different functions can have the same image.

Definition 2 *Two curves, $\gamma_1 : [a, b] \rightarrow R^n$ and $\gamma_2 : [c, d] \rightarrow R^n$, are equivalent if there is a differentiable function $\phi : [a, b] \rightarrow [c, d]$ such that $\phi' > 0$, $\phi(a) = c$, $\phi(b) = d$, and*

$$\gamma_1(t) = \gamma_2(\phi(t))$$

for $a \leq t \leq b$.

This implies that $\gamma_1([a, b]) = \gamma_2([c, d])$, so the images are the same. Also, they go in the same direction, or have the same “orientation”:

Definition 3 *An “orientation” for a smooth curve $\gamma : [a, b] \rightarrow R^n$ is a smooth function $T : [a, b] \rightarrow R^n$ such that for each $t \in [a, b]$, $T(t)$ is a unit vector, and $T(t) = c\gamma'(t)$ for some $c \neq 0$. (c depends on t .)*

Notice that $\gamma'(t) \neq 0$, from the definition of a smooth curve. Note also that if $\gamma(t) = (x(t), y(t))$, then $\gamma'(t) = (x'(t), y'(t))$. Each curve has two orientations, one in the direction of $\gamma'(t)$ and one in the opposite direction. Continuity of the mapping T means that it cannot suddenly switch directions. Since it is a unit vector, it has to keep always in the same direction, so that either $T(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}$ for all $t \in [a, b]$ or $T(t) = -\frac{\gamma'(t)}{\|\gamma'(t)\|}$ for all $t \in [a, b]$. We will later consider oriented surfaces, a more difficult concept.

Given a smooth curve $\gamma(t) = (x(t), y(t))$, $a \leq t \leq b$, in R^2 , and a continuous vector valued function $F = (f, g) : R^2 \rightarrow R^2$, we define the “line integral of F over γ ”, written as

$$\int_{\gamma} f dx + g dy,$$

to be

$$\int_{\gamma} f dx + g dy := \int_a^b (f, g) \cdot \gamma'(t) dt = \int_a^b f(x(t), y(t)) x'(t) dt + \int_a^b g(x(t), y(t)) y'(t) dt$$

If we think of $(f(x, y), g(x, y))$ as a force exerted on a particle at point (x, y) , then $\int_{\gamma} f dx + g dy$ is the work done by this force on the particle as it moves along the curve (or minus the work done by the particle in order to move). Note that the integrals on the right side are standard one dimensional integrals, as discussed in Chapter 4 of the text.

1.1 The length of a smooth curve, and reparametrization.

Assume that $\gamma : [a, b] \rightarrow R^n$ is a smooth curve. Then the “length” of γ is defined to be

$$L(\gamma) = \int_a^b |\gamma'(t)| dt. \tag{1}$$

The motivation for this definition is given in elementary calculus books. For it to be consistent with our ordinary idea of length, it first ought to give the right answer for curves which are straight lines, and second, it should be the same for all curves with the same image set, points of which are encountered in the same order. We get curves with the same image set by “reparametrizing”. This means to make a substitution $t = \tau(r)$ in the integral (1). We will consider reparametrizations in which τ' is continuous, but piecewise continuous is good enough. It is also required that $\tau' > 0$, so $\rho := \tau^{-1}$ exists. Thus, $\tau(\rho(t)) = t$ for $a \leq t \leq b$. Observe that

$$\int_a^b |\gamma'(t)| dt = \int_{\rho(a)}^{\rho(b)} |\gamma'(\tau(r))| \tau'(r) dr.$$

As an example, let $\gamma(t) = (t, t^2)$ for $1 \leq t \leq 2$. Then $\gamma'(t) = (1, 2t)$. We find that

$$s(\gamma) = \int_1^2 |\gamma'(t)| dt = \int_1^2 \sqrt{1 + 4t^2} dt \tag{2}$$

Define a second curve by $\delta(r) = (r^{1/3}, r^{2/3})$ for $1 \leq r \leq 8$. Then δ and γ have the same image sets, which is the part of the parabola $y = x^2$ lying between $x = 1$ and $x = 2$. Using $\delta'(r) = (\frac{1}{3}r^{-2/3}, \frac{2}{3}r^{-1/3})$ we can compute the length of δ :

$$L(\delta) = \int_1^8 |\delta'(r)| dr = \int_1^8 \sqrt{\left(\frac{1}{9}r^{-4/3} + \frac{4}{9}r^{-2/3}\right)} dr \quad (3)$$

This is the same as the integral on the right in (2), which is seen by making the substitution $r = t^3$, $dr = 3t^2 dt$ in (3) and getting (2). So the lengths of δ and γ are the same – which they should be.

1.2 Parametrize with respect to arc length

One reparametrization is particularly important. Recall that the arclength of a smooth curve $\gamma : [a, b] \rightarrow R^n$ is given by

$$L = \int_a^b |\gamma'(t)| dt.$$

If we let

$$s(t) = \int_a^t |\gamma'(\sigma)| d\sigma,$$

then s defines a reparametrization of γ , and the arclength is simply

$$L = \int_{\gamma} ds$$

Also, suppose that $f : R^n \rightarrow R$ is continuous. Then we define $\int_{\gamma} f ds$ by

$$\int_{\gamma} f ds = \int_a^b f(\gamma(t)) |\gamma'(t)| dt. \quad (4)$$

2 Differential forms from R^2 to R .

2.1 Definition of a differential form

In a sense, the simplest “differential forms” are simply a different notation for line integrals of this type. A generalization applies to “surface integrals”, which we will discuss later. But I think it is useful to give something more concrete to think of as a differential form than just a “notation”, whatever that is. For the moment we will stick with differential forms on R^2 .

Definition 4 A differential form on a set $U \subset \mathbb{R}^2$ is a mapping ω such that for each $x \in U$, $\omega(x)$ is a linear function from \mathbb{R}^2 to \mathbb{R} .

In other words, for each $x \in U$, $\omega(x) \in L(\mathbb{R}^2, \mathbb{R})$.

Example 5 Suppose that $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, and f is differentiable at every $x = (x_1, x_2) \in \mathbb{R}^2$. Then Df is a differentiable form on $U = \mathbb{R}^2$. As we saw in Chapter 6,

$$\begin{aligned} Df(x_1, x_2) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \left(\frac{\partial f}{\partial x_1}(x_1, x_2), \frac{\partial f}{\partial x_2}(x_1, x_2) \right) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ &= f_1 u_1 + f_2 u_2, \end{aligned}$$

where now I am using the notation $\frac{\partial f}{\partial x_1} = f_1$, $\frac{\partial f}{\partial x_2} = f_2$.

Example 6 Let $\omega(x_1, x_2) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (x_1^2 x_2, x_1 x_2^2) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = x_1^2 x_2 u_1 + x_1 x_2^2 u_2$.

We can ask whether this is a special case of the previous example. In other words, is there a function $f(x_1, x_2)$ such $f_1(x_1, x_2) = x_1^2 x_2$, $f_2(x_1, x_2) = x_1 x_2^2$? It is not hard to see that there is no such function f , for if there were, then the second partial derivatives (which are clearly continuous) would have to satisfy $f_{12} = f_{21}$, and that is not true, because $\frac{\partial(x_1^2 x_2)}{\partial x_2} = x_1^2$, while $\frac{\partial(x_1 x_2^2)}{\partial x_1} = x_2^2$.

Definition 7 A differential form $\omega : U \rightarrow \mathbb{R}$ is called “exact” if there is a differentiable function $f : U \rightarrow \mathbb{R}$ such that $\omega = Df$.

Thus, the previous example is not exact.

Example 8 I’ll switch to (x, y) notation. Let $U = \mathbb{R}^2 \setminus \{(0, 0)\} = \{(x, y) \mid (x, y) \neq (0, 0)\}$.

For $(x, y) \in U$, let $\omega(x, y) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = - \left(\frac{xu_1}{(x^2+y^2)^{3/2}}, \frac{yu_2}{(x^2+y^2)^{3/2}} \right)$. It is not hard to show

that if $f(x, y) = \frac{1}{\sqrt{x^2+y^2}}$, then $\omega \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (f_x, f_y) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. That is, $\omega = Df$. Hence ω is exact. Notice that ω is the gravitational force vector (in the right units!), since $\|\omega\| = \frac{1}{x^2+y^2}$.

Example 9 With the same U , let $\omega(x, y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$. You can check that

$\frac{\partial}{\partial y} \left(\frac{-y}{x^2+y^2} \right) = \frac{\partial}{\partial x} \left(\frac{x}{x^2+y^2} \right)$. Thus, we would expect there to be an f such that $\omega = Df$.

However, using line integrals, we will show later that there is no such f , and so ω is not exact.

2.2 Standard notation for differential forms.

We wish to relate differential forms to line integrals. Suppose that $\omega(x, y) \begin{pmatrix} u \\ v \end{pmatrix} = p(x, y)u + q(x, y)v$. We can write this in vector form:

$$\omega(x, y)(u, v) = (p(x, y), q(x, y)) \begin{pmatrix} u \\ v \end{pmatrix}.$$

It is tempting to say: ω is a gradient. However this is only true if ω is exact.

How does this relate to line integrals? The key idea is to switch notation again, and use the symbols “ dx ” for u and “ dy ” for v . So the expression

$$x^2ydx + y^2xdy$$

can be considered to be a differential form. Often one sees in books something like this:

A differential form is an expression of the form

$$p(x, y) dx + q(x, y) dy. \tag{5}$$

We have to realize that here “ dx ” is an expression for a single variable, say u , which could have any value. There is no implication that somehow dx represents a small change in a variable x .

(Nevertheless, in many applications it may be useful at times to think of dx in this way.)

So the relation of differential forms to line integrals may be considered purely notational. Suppose that $\gamma : R \rightarrow R$ is a smooth curve, say $\gamma(t) = (x(t), y(t))$ for $a \leq t \leq b$. Suppose that ω is a differential form, with $\omega(x, y) \begin{pmatrix} u \\ v \end{pmatrix} = p(x, y)u + q(x, y)v$. Then we define in line integral of ω over γ , denoted by

$$\int_{\gamma} \omega$$

to be

$$\int_{\gamma} \omega = \int_{\gamma} p dx + q dy := \int_a^b p(x(t), y(t)) x'(t) dt + \int_a^b q(x(t), y(t)) y'(t) dt. \tag{6}$$

3 Differential forms on R^3 .

This discussion is easily generalized to R^n . But when $n \geq 3$ there is more than one useful kind of differential form. The one we just described is called a “one-form”. A differential one-form ω on an open set $U \subset R^n$ is a map $\omega : R^n \rightarrow L(R^n, R)$, which we have seen is equivalent to an ordered set of n functions $(a_1(x), a_2(x), \dots, a_n(x))$. One example is when $\omega = Df$ for some smooth f , in which case, ω can be considered to be a gradient: $\omega = \text{grad } f$ for some f , but not all differential forms are of this type, only (by definition), those which are “exact”.

But on R^n there are more complicated differential forms. We will stick with “two-forms”.

We earlier discussed the second derivative $D^2f(x)$ of a smooth function f at a point x . We said that it was a “bilinear” map from $R^n \times R^n$ to R . This means that $D^2f(x)(u, v)$ is linear in each variable, u or v , separately.

One thing we didn’t address is the following: Does $D^2f(x)(u, v) = D^2f(x)(v, u)$? Let’s try an example for $n = 2$. Say that the standard matrix corresponding to $D^2f(x)$ is $A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$. Then

$$D^2f(x)(u, v) = u^T Av$$

Take the transpose, recognizing that $A = A^T$. Since $D^2f(x)(u, v)$ is one dimensional, it is equal to its transpose, giving

$$D^2f(x)(u, v) = (D^2f(x)(u, v))^T = (u^T Av)^T = v^T A^T u = v^T Au = D^2f(x)(v, u).$$

But clearly, this only works if A is symmetric.

A general bilinear function does not require a symmetric matrix. For example,

$$L(u, v) := u^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} v = u_1 v_2 - u_2 v_1$$

is bilinear. However, $L(u, v) = -L(v, u)$. A function $L : R^n \times R^n \rightarrow R$ satisfying this relation is called an “alternating bilinear” function. It also is special, and most bilinear functions will be neither “commuting” nor alternating.

Definition 10 A “two-form” on an open set $U \subset R^n$ is a map $\omega : U \rightarrow L(R^n \times R^n, R)$ such that for each $x \in U$, $\omega(x)$ is an alternating bilinear function from $R^n \times R^n$ to R .

Example 11 For $n = 3$, let $U = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. For $(x, y, z) \in U$, let $\omega(x, y, z)(u, v) = \det \begin{pmatrix} u_1 & v_1 \\ u_3 & v_3 \end{pmatrix}$. Thus,

$$\omega(x)(u, v) = u_1 v_3 - v_1 u_3.$$

this is clearly an alternating bilinear function, which in this case doesn't depend on x . We can define many other such functions on R^3 , using the fact that the sign of a determinant changes when you interchange two columns.

Just as one-forms define line integrals, two-forms define surface integrals in R^3 . We shall return to this topic in due course.

4 Homework – second set of problems due on February 23.

4. Suppose that α and β are equivalent curves and ω is a continuous differential form. Prove that

$$\int_{\alpha} \omega = \int_{\beta} \omega.$$

(The definition of $\int_{\alpha} \omega$ is in equation (6) above.)

5. Let γ be a smooth curve, with orientation T given by

$$T(t) = \frac{\gamma'(t)}{|\gamma'(t)|}.$$

Also, if $\gamma = (\gamma_1, \gamma_2)$, let

$$N(t) = \frac{1}{|\gamma'(t)|} (\gamma_2'(t), -\gamma_1'(t)).$$

(a) Prove that for each t , $N(t)$ is a unit normal to the curve at $\gamma(t)$.

(b) Prove that if $\omega = -F_2 dx + F_1 dy$, then

$$\int_{\gamma} F \cdot N ds = \int_{\gamma} \omega.$$

(See (4) above for the definition of $\int_{\gamma} F \cdot N ds$. Set $f = F \cdot N$. Note that $f : R^n \rightarrow R$, as is needed for $\int_{\gamma} f ds$ to be defined.) Also, see (6) for the definition of $\int_{\gamma} \omega$.