Math 1540
Analysis II
Spring, 2011
Notes #1

These notes are supplementary to the text, and do not replace it. I will post them on the website, at irregular intervals. Not all material in the text, or in the lectures, will be discussed in the notes. Ordinarily, you should read the text first, then the relevant notes, if any. This first set is concerned with the material in Sections 5.5 and 5.6 of the text.

1 The space of continuous functions

While you have had rather abstract definitions of such concepts as metric spaces and normed vector spaces, most of 1530, and also 1540, are about the spaces R^n. This is what is sometimes called "classical analysis", about finite dimensional spaces, and provides the essential background to graduate analysis courses. More and more, however, students are being introduced to some infinite dimensional spaces earlier. This section is about one of the most important of these spaces, the space of continuous functions from some subset A of a metric space M to some normed vector space N. The text gives a careful definition, calling the space $C(A, N)$.

The simplest case is when $M = R$ $(= R^1)$. Let A be a subset of R. Then let

$$C(A, R) = \{f : A \to R \mid f \text{ is continuous}\}.$$

In the most common applications A is a compact interval. Thus $C([0,1], R)$ is the space of all continuous $f : [0,1] \to R$.

When $A \subset R$ and $N = R$, $C(A, R)$ is often shortened to $C(A)$. However any careful writer makes it clear what is meant by this notation. More generally, A could be a subset of any metric space M, and f could map M to R or to R^m.

Definition 1 *If $A \subset M$, then*

$$C(A, R^m) = \{f : A \to R^m \mid f \textit{ is continuous}\}.$$

It is easy to turn $C(A, R^m)$ into a vector space, by defining addition and scalar multiplication in the usual way: $(f + g)(x) := f(x) + g(x)$, $(cf)(x) = c(f(x))$. You probably ran into this even in linear algebra. (I use $:=$ to denote an equation which defines the quantity on the left.)

However it is also very useful to define a norm, so that $C(A, R^m)$ is a normed vector space. This is a little trickier. We shall only do so when A is a compact set. In that case, Theorem 4.2.2 (in the text) implies that each $f \in C(A, R^m)$ is bounded. (It is a good thought exercise to look at Theorem 4.2.2, observe that the word "bounded" does not appear, and explain to yourself why it implies that in the case we are considering, f is bounded.) This allows us to define a norm. With an eye toward the application we will give shortly, we take $m = 1$.

Definition 2 *If $A \subset R$ is compact, and $f \in C(A, R)$, let*

$$||f|| = \max_{x \in A} |f(x)|.$$

Note that in this definition I am assuming that the "maximum" of f is defined. This is justified by the Maximum-Minimum theorem, Theorem 4.4.1. If I say that the maximum of a function f is c, this means that there is a particular x in the domain of f such that $f(x) = c$, and furthermore, there is no x with $f(x) > c$. Be sure you are straight on the difference between a maximum and supremum.

The text gives the definition required for more general A and N. As an example, consider $C(I, R)$ where I is the interval $(-\infty, \infty)$. In that case $f : I \to R$ can be continuous and yet not have a maximum. You should be able to think of an example which is a bounded function. More significantly, f might not be bounded.

In that case, $C(I, R)$ is not turned into a normed space in a natural way. (There may be some peculiar way to do it.) Instead, for general A, the text considers

$$C_b(A, R) := \{f \in C(A, R) \mid f \text{ is bounded}\}.$$

Note that this is a vector space. Then for $f \in C_b(A, R)$ we let

$$||f|| = \sup_{x \in A} |f(x)|.$$

(What theorem tells us this exists?) Some homework at the end of the section is concerned with this.

Theorem 5.5.1 implies that if A is compact then $C(A, R)$ is a normed linear space, as is $C_b(A, R)$ for general A. We will mainly be concerned with part (ii) of this theorem, since I think that in all our examples N will be a normed vector space.

Once we define a norm, we can define convergence of sequences.

Definition 3 *A sequence $\{f_k\} \subset C_b(A, R)$ converges to $f \in C_b(A, R)$ if*

$$\lim_{k \to \infty} ||f_k - f|| = 0.$$

As the text points out, the following assertion is then obvious from the definitions of convergence of a sequence in $C_b(A,R)$ and uniform convergence as it was discussed in section 5.1.

Proposition 4 *A sequence* $\{f_k\} \subset C_b(A,R)$ *converges to* $f \in C_b(A,R)$ *if and only if*

$$\lim_{k\to\infty} f_k(x) = f(x)$$

uniformly on A.

Why? and does it make any difference if A is compact? If A is compact, what is the relation between $C(A,R)$ and $C_b(A,R)$? What can you say about these two spaces if you don't know that A is compact (it might be, or not).

2 Examples

1. $n = 1$, $A = [0,1]$. The space $C([0,1],R)$ is often denoted by $C([0,1])$, the image space R being understood. Clearly, $f(x) = x^2 + \cos x + e^x$ is in $C(A,R)$, but

$$g(x) = \begin{cases} \frac{1}{x} \text{ if } 0 < x \leq 1 \\ 0 \text{ if } x = 0 \end{cases}$$

is not.

2. $n = 1$, $A = (0,1)$. Then $g(x) = \frac{1}{x}$ for $0 < x < 1$ is in $C(A,R)$. Is g in $C_b(A,R)$?

3. $n = 2$, $A = [0,1] \times [0,\infty)$. Is x^y in $C_b(A,R)$? What about y^x ?

3 The Ascoli-Arzela theorem

This is one of the most important theorems in analysis. As one application, it tells us what subsets of $C(A,R)$ are compact.

First let's recall how to decide if a subset S of R^n is compact. Since R^n is a metric space, the Bolzano-Weierstrass theorem tells us that S is compact if and only if every sequence of points in S has a convergent subsequence. Then the Heine-Borel theorem says that a set R^n is compact if and only if it is closed and bounded.

We saw that $C([0,1])$ is a normed linear space. Let

$$S = \{f \in C([0,1]) \mid ||f|| \leq 1\}.$$

Obviously S is bounded. To see that S is closed, suppose that $\{f_n\} \subset S$ and $f_n \to g$ in $C([0,1])$. (This was defined in the previous section.) I claim that $||g|| \leq 1$.

Suppose not, and $||g|| > 1$. Let $\delta = ||g|| - 1$, or

$$||g|| = \delta + 1 \tag{1}$$

Then there is an N such that for $n \geq N$, $||f_n - g|| < \frac{\delta}{2}$. In this case, by the triangle inequality,

$$||g|| = ||g - f_n + f_n|| \leq ||g - f_n|| + ||f_n|| \leq \frac{\delta}{2} + 1,$$

which contradicts (1).

Hence, S is closed and bounded. Let us see if S is compact. Consider the sequence $\{f_n\}$ where $f_n(x) = x^n$ for $0 \leq x \leq 1$. Then $f_n \in S$ for each n. Also.

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 \text{ if } 0 \leq x < 1 \\ 1 \text{ if } x = 1. \end{cases}$$

Hence,$\{f_n\}$ does not converge to a continuous function. Further, no subsequence of $\{f_n\}$ converges to a continuous function. (Do you think this needs further proof? If so, fill in the details!)

This shows that S is not compact. The Heine-Borel theorem does not hold in the infinite dimensional normed linear space $C([0,1])$. Some other testable condition is needed to guarantee that a set S is compact.

Definition 5 *A sequence $\{f_n\}$ of functions in $C([0,1])$ is uniformly bounded if there is an $M > 0$ such that $||f_n|| \leq M$ for $n = 1, 2, 3, \cdots$.*

Definition 6 *A sequence S of functions in $C([0,1])$ is equicontinuous if for each $\varepsilon > 0$ there is a $\delta > 0$ such that if x and y are in $[0,1]$ with $|x - y| < \delta$, and $f \in S$, then*

$$|f(x) - f(y)| < \varepsilon. \tag{2}$$

In other words, in the definition of continuity of a function, the δ does not depend on x, y or the function f chosen from S. (Compare this with the definition of uniform continuity of a function.)

One example of an equicontinuous set of functions is a set S of functions in $C([0,1])$ such that every function f in S satisfies a Lipschitz condition in $[0,1]$ and the Lipschitz constant L can be chosen to be the same for every f in S. In this case, if $\varepsilon > 0$ is given then choose $\delta = \frac{\varepsilon}{L}$ and check that (2) holds. The δ does not depend on f.

For instance, we can let S be the sequence $\{f_n\}$ of functions given by $f_n(x) = \frac{x^n}{n}$. Then $f_n'(x) = x^{n-1}$, so $|f_n'(x)| \leq 1$ for every n and every $x \in [0,1]$. By the mean value theorem we can choose $L = 1$.

On the other hand, we will show that $\{x^n\}$ is a uniformly bounded sequence (on $[0,1]$), but not an equicontinuous one. Uniform boundedness is obvious. For equicontinuity, suppose that $\varepsilon = \frac{1}{2}$ and that there is a δ such that (2) holds for every n and every x and y in $[0,1]$ with $|x-y| < \delta$. Choose $y = 1$ and $x = 1 - \frac{\delta}{2}$. Since $\lim_{n\to\infty}\left(1-\frac{\delta}{2}\right)^n = 0$, we can choose n so large that $f_n(1) - f_n\left(1-\frac{\delta}{2}\right) = 1 - \left(1-\frac{\delta}{2}\right)^n > \frac{1}{2}$, contradicting (2). Hence, S is not equicontinuous. As we saw, S does not have a convergent subsequence.

Theorem 7 *(Ascoli-Arzela) If $S = \{f_n\}$ is a uniformly bounded and equicontinuous sequence of functions in $C([0,1])$, then S has a subsequence which converges in $C([0,1])$.*

Remark 8 *It is important that the limit function is also in $C([0,1])$. But this is what we mean when we say that a subsequence "converges in $C([0,1])$."*

The proof (for a more general case) is in the text, and will be given in class.

The practical application of this theorem is often easier than the example above might suggest. If f is differentiable on $[0,1]$, and for some , $|f'(x)| \leq L$ for every $x \in [0,1]$, then L is a Lipschitz constant for f. This follows from the mean value theorem. Hence, for a sequence $\{f_n\}$, if each f_n is differentiable on $[0,1]$, and $|f_n'(x)| \leq L$ for every n and every $x \in [0,1]$, then $\{f_n\}$ is equicontinuous. If in addition, there is a K such that $||f_n|| \leq K$ for every n, then $\{f_n\}$ is equicontinuous and uniformly bounded, and some subsequence of $\{f_n\}$ converges in $C([0,1])$.

Turning to compactness, if $S \subset C([0,1])$, every $f \in S$ is differentiable on $[0,1]$, and there are numbers L and B such that $||f|| \leq B$ and $|f'(x)| \leq L$ for every $f \in S$ and every $x \in [0,1]$, then S is a compact subset of $C([0.1])$.

As an example of a compact set we can consider

$$S = \left\{ g \mid g(x) = \int_0^x f(s)\,ds \text{ for some } f \in C([0,1]) \text{ with } ||f|| \leq 1 \right\}.$$

You should use the Ascoli-Arzela theorem to show that S is compact.

4 Contraction mapping theorem

This is another result that can be used to show that sequences converge. It is easier to prove than Ascoli-Arzela and gives a stronger result. However there are many examples where Ascoli-Arzela can be applied but the contraction mapping theorem cannot. The statement below is weaker than the one in the text, but is adequate for many applications, including the one given in the next section. You should compare it with the one in the text (Theorem 5.7.1) and figure out what makes the text version "stronger". Also determine why the proof given below would not be a valid proof for Theorem 5.7.1.

Theorem 9 *Let B be a Banach space, M a closed subset of B, and Φ a mapping from M to M such that for some $k \in [0,1)$,*

$$||\Phi(x) - \Phi(y)|| \leq k\,||x-y||$$

for any two points x and y in M. Then there is a unique point z in M such that $\Phi(z) = z$.

Definition 10 *Such a point z is called a "fixed point" for Φ.*

Proof. Choose some x_0 in M. Define a sequence $\{x_i\}$ by setting $x_{i+1} = \Phi(x_i)$, for $i = 0, 1, 2, \cdots$. Then for any n,

$$x_n = x_0 + (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}).$$

Also, for $i \geq 1$,

$$||x_{i+1} - x_i|| \leq k\,||x_i - x_{i-1}||,$$

and by induction we easily show that

$$||x_{i+1} - x_i|| \leq k^i\,||x_1 - x_0||.$$

Because $|k| < 1$, $\Sigma_{i=1}^{\infty} k^i$ converges, which implies that $\Sigma_{i=1}^{\infty} ||x_{i+1} - x_i||$ converges. (Why?) By the Weirerstrass M test (Theorem 5.2.2), $\Sigma_{i=1}^{\infty}(x_{i+1} - x_i)$ converges in B, and hence $\lim_{n\to\infty} x_n$ exists. Let z be this limit. Since M is closed and $x_n \in M$ for each n, $z \in M$. Also, $x_{n+1} = \Phi(x_n)$, and so (from the definition of limit)

$$\lim_{n\to\infty} \Phi(x_n) = \lim_{n\to\infty} x_{n+1} = z.$$

Further, for any n

$$\begin{aligned}\|\Phi(z) - z\| &= ||\Phi(z) - \Phi(x_n) + \Phi(x_n) - z|| \\ &\leq k\,||z - x_n|| + ||\Phi(x_n) - z||.\end{aligned}$$

Since the limit of the right side as $n \to \infty$ is zero, and the left side is independent of n, it follows that the left side is zero for every n, and so z is a fixed point for Φ.

To prove uniqueness, suppose that there are two fixed points, say x and y. Then

$$f(x) = x, \;\; f(y) = y$$

and so

$$|x - y| = |f(x) - f(y)| \leq k\,|x - y|$$

where $0 < k < 1$. This is only possible if $x = y$. ■

5 Application

There is an easy application of the contraction mapping theorem to differential equations. We consider an "initial value problem" for an ode, of the form

$$x' = f(t, x) \tag{3}$$

$$x(0) = 0. \tag{4}$$

Here, x is one dimensional. It is also possible to consider systems of ode's, in which x and $f(t, x)$ are n-dimensional, but we will stick with one dimensional case.

Theorem 11 *Suppose that f is continuous in a rectangle Ω given by $-T \leq t \leq T$, $-X \leq x \leq X$. Suppose also that in Ω the function f satisfies a Lipschitz condition of the form*

$$|f(t, x) - f(t, y)| \leq K\,|x - y|\,.$$

Then there is some $\delta > 0$ such that the initial value problem (3)-(4) has a unique solution $\phi(t)$ on the interval $[-\delta, \delta]$ such that $\phi(0) = 0$.

Remark 12 *Notice that I have used ϕ to denote a solution to the ode (3). This is sometimes more convenient than denoting the solution by x. To say that ϕ is a solution to (3) is simply to say that $\phi'(t) = f(t, \phi(t))$.*

Remark 13 *It is important also to note that the interval $[-\delta, \delta]$ may be smaller than $[-T, T]$. In class I will give an example showing that this is a necessary restriction in the theorem. The solution may not exist on $[-T, T]$.*

Proof. We will use an integral equation which is equivalent to the initial value problem (3)-(4). If ψ is a continuous function on the interval $I := [-T, T]$, with $|\psi(t)| \leq X$ on I, and f is continuous on Ω, then Theorem 4.3.1 implies that $f(s, \psi(s))$ is a continuous function of s for $s \in I$. Then Theorem 4.8.4 implies that $\int_0^t f(s, \psi(s))\, ds$ is defined for $s \in I$, and the fundamental theorem of calculus tells us that this function is differentiable and that

$$\frac{d}{dt}\left(\int_0^t f(s, \psi(s))\, ds\right) = f(t, \psi(t)).$$

Let

$$L = \max_{(t,x)\in\Omega} |f(t, x)|$$

and set

$$\delta = \min\left\{T, \frac{X}{L}, \frac{1}{2K}\right\},$$

Then let $B = C([-\delta, \delta])$ and $M = \{\psi \in B \mid ||\psi|| \leq X\}$, where $||\psi|| = \max_{-\delta\leq t\leq\delta} |\psi(t)|$. It is easy to show that M is a closed subset of B. On M define a mapping $\Phi : M \to B$ by

$$\Phi(\psi)(t) = \int_0^t f(s, \psi(s))\, ds.$$

If $\psi \in M$, then for each t with $|t| \leq \delta$ we use the definition of δ to get

$$||\Phi(\psi)|| \leq \left|\int_0^t L ds\right| = |Lt| \leq L\delta \leq X.$$

Hence, $\Phi : M \to M$. Also, if ψ_1 and ψ_2 are in M, then for each $t \in [-\delta, \delta]$,

$$\begin{aligned}|\Phi(\psi_1)(t) - \Phi(\psi_2)(t)| &\leq \int_0^t |f(s, \psi_1(s)) - f(s, \psi_2(s))|\, ds \leq \int_0^\delta |f(s, \psi_1(s)) - f(s, \psi_2(s))|\, ds \\ &\leq \int_0^\delta K|\psi_1(s) - \psi_2(s)|\, ds \leq K\delta\, ||\psi_1 - \psi_2||.\end{aligned}$$

Hence $||\Phi(\psi_1) - \Phi(\psi_2)|| \leq K\delta\, ||\psi_1 - \psi_2||$. Since $K\delta < 1$, Φ is a contraction on M, and so by the contraction mapping theorem, Φ has a unique fixed point, which we denote by ϕ. Then for $|t| \leq \delta$,

$$\phi(t) = \int_0^t f(s, \phi(s))\, ds. \tag{5}$$

Obviously $\phi(0) = 0$, and by differentiating (5) we see that ϕ satisfies (1). This proves Theorem 11. ■

6 Homework (Due Wednesday, January 12 at the beginning of class. No late homework accepted. Problems will be discussed at the beginning of the class where they are due.)

When the answer is in the back of the book, you still have to prove that it is correct.

1. pg. 272, # 1
2. pg. 272, # 4
3. pg. 275, #5.
4. pg. 282, # 4.
5. pg. 322, # 47 (first part). (You can take $m = n = 1$.)