Math 1280

Notes 9a

A basic example of the use of the Delta method

In this example we redo an example treated earlier by standard series substitution. This was the equation

$$2xy'' + y' + xy = 0. \tag{1}$$

Recall that the $\delta$ operator is defined by

$$\delta y = xy'$$

In other words,

$$(\delta y)(x) = x'y'(x).$$

From this we get

$$\begin{aligned}\delta^2 y &= \delta(xy') = x(xy')' = x(xy'' + y') \\ &= x^2y'' + xy',\end{aligned}$$

or

$$x^2y'' = \left(\delta^2 - \delta\right) y.$$

To use this with (1) we need to mulitply (1) by $x$, to give

$$2x^2y'' + xy' + x^2y = 0.$$

We then express this in terms of $\delta$, putting terms which still have an $x$ on the right.

$$\left(2\left(\delta^2 - \delta\right) + \delta\right) y = -x^2y.$$

Simplifying,

$$\left(2\delta^2 - \delta\right) y = -x^2y.$$

Now write

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

Use the formulas

$$\begin{aligned}\delta(x^p) &= x(x^p)' = xpx^{p-1} = px^o, \\ \delta^2(x^p) &= p^2x^p\end{aligned}$$

and get

$$\sum_{n=0}^{\infty}\left(2\left(n+r\right)^2-\left(n+r\right)\right)a_n x^{n+r}=-\sum_{n=0}^{\infty}a_n x^{n+r+2}=-\sum_{n=2}^{\infty}a_{n-2}x^{n+r}.$$

Putting everything on the left, we give the first two terms and then the general term:

$$\left(2r^2-r\right)a_0x^r+\left(2\left(1+r\right)^2-\left(1+r\right)\right)a_1x^r$$
$$+\sum_{n=2}^{\infty}\left\{\left(2\left(n+r\right)^2-\left(n+r\right)\right)a_n+a_{n-2}\right\}x^{n+r}=0.$$

As we did before, we choose $r$ so that $2r^2-r=0$, then set $a_1=0$ and

$$a_n=-\frac{a_{n-2}}{2\left(n+r\right)^2-\left(n+r\right)}$$

for $n\geq 2$. To get specific solutions, set $a_0=1$. For $r=0$ we get

$$1-\frac{1}{6}x^2+\frac{1}{6\left(28\right)}x^4+\cdots.$$

For $r=\frac{1}{2}$ we get

$$x^{\frac{1}{2}}\left(1-\frac{1}{10}x^2+\cdots\right).$$