Math 1280
Notes 9
"Delta method" for regular singular points

We now consider how to solve an equation with a "regular singular point". The method to be given is different from that in the text, and better! I will try to verify that claim by doing some of the same examples as in the text, by this different method.

We are considering equations

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0 \tag{1}$$

where P, Q, R are polynomials. For simplicity, we will assume that the singular point is at $x = 0$. It is then easy to say what is meant by a regular singular point:

Definition: *The point* $x_0 = 0$ *is a regular singular point for* (1) *if* $P(0) = 0$, *at least one of* $Q(0)$ and $R(0)$ is not zero, *and the functions*

$$\frac{xQ(x)}{P(x)} \text{ and } \frac{x^2R(x)}{P(x)}$$

are continuous at $x = 0$.

This has to be understood as meaning that $\lim_{x\to 0}\frac{xQ(x)}{P(x)}$ and $\lim_{x\to 0}\frac{x^2R(x)}{P(x)}$ both exist. Therefore, both expressions can be defined at $x = 0$ in such a way as to make them each continuous.

Here are some examples:

1. The equation

$$\left(1+x^2\right)y'' + xy' + y = 0$$

does not have any sort of singular point at $x = 0$, since $P(0) = 1$

2. The equation

$$x^2y'' + xy' + y = 0$$

has a regular singular point, because if $x \neq 0$, then

$$\frac{xQ(x)}{P(x)} = \frac{x^2}{x^2} = 1, \quad \frac{x^2R(x)}{P(x)} = \frac{x^2}{x^2} = 1 \tag{2}$$

Thus both of the limits above exist.

3. The equation

$$x^3y'' + xy' + xy = 0$$

has an **irregular** singular point at 0, because

$$\frac{xQ(x)}{P(x)} = \frac{x^2}{x^3} = \frac{1}{x},$$

which cannot be made continuous at $x = 0$. Note that $\frac{x^2R(x)}{P(x)} = 1$, which is ok, but that is not enough to rescue the equation.

First we wish to get an idea of what sort of series to expect. The simplest examples of regular singular points are those discussed in section 5.5, called "Euler equations". We saw in class that they usually have solutions of the form x^r, where r is some number which satisfies a quadratic equation in r. But they may have terms involving logarithms.

Here is an example which is not an Euler equation: (problem 1, pg. 284)

$$2xy'' + y' + xy = 0.$$

It is easily checked that this has a regular singular point at $x_0 = 0$. Since it is not an Euler equation, we don't expect x^r to be a solution. Instead, we look for a solution

$$y = x^r \sum_{n=0}^{\infty} a_n x^n. \tag{3}$$

Before taking derivatives, we bring the x^r inside the summation.

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2} ...$$

Notice that we can't drop any terms at the beginning, because we can't assume that that $(n+r)$ or $(n+r-1)$ are zero. Substituting into our differential equation gives

$$2\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0.$$

Now we factor the x^r out again:

$$x^r \left(2\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n-1} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} \right)$$

We can divide by x^r.

We should be careful in shifting indices. In the first two terms, let $m = n - 1$, so $n = m + 1$. In the last term, we let $m = n + 1$. Doing this, and then changing back to n, gives

$$\sum_{n=-1}^{\infty} 2\left(n+1+r\right)\left(n+r\right)a_{n+1}x^n + \sum_{n=-1}^{\infty}\left(n+1+r\right)a_{n+1}x^n + \sum_{n=1}^{\infty} a_{n-1}x^n$$

The lowest power of x is now x^{-1}. We get

$$\begin{aligned} x^{-1}\left(2r\left(r-1\right)+r\right)a_0 &= 0 \\ x^0\left(2\left(r+1\right)r+\left(r+1\right)\right)a_1 &= 0 \\ a_{n+1} = -\frac{a_{n-1}}{\left(n+r+1\right)\left(2\left(n+r\right)+1\right)} &= -\frac{a_{n-1}}{\left(n+r+1\right)\left(2n+2r+1\right)}. \end{aligned}$$

The first equation simplifies to

$$x^{-1}\left(2r^2-r\right)a_0 = 0$$

while the second is

$$\left(2r^2+3r+1\right)a_1 = 0$$

This presents a bit of a dilemma. We still have to determine r. We can get zero from the first equation by solving $2r^2 - r = 0$, which gives $r_1 = 0$, $r_2 = \frac{1}{2}$. Neither of these give zero in the second equation, so if we have $a_0 \neq 0$, then we must have $a_1 = 0$. Or, we could set $a_0 = 0$ and solve the second equation for r :

$$\begin{aligned} 2r^2+3r+1 &= 0 \\ r &= \frac{-3 \pm \sqrt{9-8}}{4} = -1, -\frac{1}{2}. \end{aligned}$$

These differ by one from r_1 and r_2. But look at the first terms we get making these two choices. If we set $a_0 = 1, a_1 = 0$, then in (3) our series start out as $x^0\left(1 + a_2x^2 + \cdots\right)$ or $x^{\frac{1}{2}}\left(1 + a_2x^2 + \cdots\right)$. ($a_2$ would be different in these two solutions.

On the other hand, if we set $a_0 = 0, a_1 = 1$, in (3) our series start as $x^{-1}\left(x^1 + a_3x^3 + \cdot\cdot\right)$ or $x^{-\frac{1}{2}}\left(x^1 + a_3x^3 + \cdots\right)$. We end up getting the same solutions in either case. Hence, we will assume that $a_1 = 0$, and find r from the coefficient of a_0. Since we get two different r's, we get two different solutions, which is what we expect for a second order linear equation.

Definition: *The equation for r obtained by setting $a_0 = 1, a_1 = 0$ is called the "indicial equation".*

1 Delta Method for series solutions

To discuss the "delta method", we need to become comfortable with "differential operators". You may have run into these before. If $y(x)$ is a differentiable function, then we let

$$Dy$$

denote the derivative of y. Thus,

$$Dy(x) = \frac{dy}{dx} = y'(x).$$

You may have seen how to use differential operators to solve second order linear equations with constant coefficients. For example,

$$y'' + 3y' + 2y = 0$$

can be written as

$$D^2y + 3Dy + 2y = 0,$$

which in turn can be written as

$$(D+2)(D+1)y = 0 \tag{4}$$

This tells us to solve

$$(r+2)(r+1) = 0$$

for r, given $r_1 = -1,\ r_2 = -2$. Then, the general solution is

$$y = c_1e^{-x} + c_2e^{-2x}.$$

In the delta method, we use another operator in a similar way. This is the operator given by

$$\delta y(x) = xDy(x) = x\frac{dy}{dx}.$$

Hence, for example, the equation

$$\delta y + 2y = 0$$

is the same as the differential equation

$$xy' + 2y = 0.$$

This can be solved using an integrating factor. Multiply by x to get

$$x^2 y' + 2xy = 0.$$

This is now in "exact" form

$$\begin{aligned} \left(x^2 y\right)' &= 0 \\ x^2 y &= c \\ y &= \frac{c}{x^2}. \end{aligned}$$

As a more general example, if we want to solve

$$\delta y + ry = 0$$

where r is constant, we write this as

$$xy' + ry = 0$$

and multiply by x^{r-1}, to get

$$\begin{aligned} x^r y' + r x^{r-1} y &= 0 \\ (x^r y)' &= 0 \\ x^r y &= c \\ y &= c x^{-r}. \end{aligned} \tag{5}$$

In order to use this the same way as we used D, to solve second order equations, we have to compute δ^2. This is a bit trickier than computing D^2, since it is obvious that $D^2 y = \frac{d^2 y}{dx^2} = y''$. To calculate $\delta^2 y$, we have

$$\begin{aligned} (\delta y)(x) &= xy'(x) \\ \left(\delta^2 y\right)(x) &= \delta\left(\delta y(x)\right) \\ &= \delta\left(xy''\right) = x\left(xy'\right)' = x\left(y' + xy''\right) \\ \left(\delta^2 y\right)(x) &= x^2 y'' + xy'. \end{aligned} \tag{6}$$

Let's see what kind of equation we get if we take the product of two operators, $(\delta + r_1 I)$ and $(\delta + r_2 I)$. We will look at a specific example, letting $r_1 = 1, r_2 = 2$. So what differential equation is given by

$$(\delta + I)(\delta + 2I)\, y = 0?$$

First of all, we will simplify the notation by writing $(\delta + r_1 I)$ as $(\delta + r_1)$. The I will be understood, since otherwise this expression would be adding "apples to oranges".

Then,as we did with the operator D in line (4) above, we can write

$$\begin{aligned}
(\delta + 2I)\, y &= \delta y + 2y = xy' + 2y \\
(\delta + 1)(\delta + 2)\, y &= (\delta + 1)(xy' + 2y) \\
&= \delta (xy' + 2y) + (xy' + 2y) \\
&= x (xy' + 2y)' + xy' + 2y \\
&= x (xy'' + y' + 2y') + xy' + 2y \\
&= x^2 y'' + 4xy' + 2y = 0.
\end{aligned}$$

Notice the extra xy' that comes in because of the product rule: $(xy')' = xy'' + y'$.

This is an Euler equation. We solve it by substituting $y = x^r$, giving

$$\begin{aligned}
(r(r-1) + 4r + 2)\, x^r &= 0 \\
r^2 + 3r + 2 &= 0.
\end{aligned}$$

This factors and gives $r_1 = -1,\ \ r_2 = -2$. The general solution is

$$y = c_1 x^{-1} + c_2 x^{-2}.$$

Thus, looking at the example ending with equation (5), we see that we get a linear combination of the solutions to $(\delta + r_1)\, y = 0$ and $(\delta + r_2 y) = 0$. This is what we would expect if we are solving $(\delta + r_1)(\delta + r_2)\, y = 0$.

(The fact that $(\delta + r_1)(\delta + r_2) = (\delta + r_2)(\delta + r_1)$ is important here. We know from matrix theory that operators do not always commute, but δ commutes with itself and with I.)

Going back to equation (6), we get another equation which is very helpful:

$$x^2 y'' = \delta^2 y - \delta y = \delta (\delta - 1)\, y$$

We could write the equations for δ and δ^2 in terms of D :

$$\begin{aligned}
xD &= \delta \\
x^2 D^2 &= \delta (\delta - 1)
\end{aligned} \tag{7}$$

Now let's use it to solve another Euler equation

$$2x^2 y'' + 3xy' - y = 0.$$

This is example 1 on page 274 of the 8th edition, 269 of the 9th edition.

From the formulas in (7) we obtain

$$2\delta\left(\delta-1\right)y+3\delta y-y=0$$

or

$$\left(2\delta^2+\delta-1\right)y=0$$

I claim that the solutions are

$$y=x^r,$$

where

$$2r^2+r-1=0.$$

To see this, we have

$$\begin{aligned}\delta x^r &= x\left(rx^{r-1}\right)=rx^r\\ \delta^2x^r &= r^2x^r\end{aligned}\tag{8}$$

and so

$$2\delta^2x^r+\delta x^r-x^r=x^r\left(2r^2+r-1\right).$$

In other words, once we get the equation in delta form, we treat δ as we did D for equations with constant coefficients, except that we use x^r for the solution instead of e^{rx}.

You will note that on page 274, in example 1, the same equation is obtained for r. The roots of

$$2r^2+r-1$$

are $r_1=\frac{1}{2}, r_2=-1$, and so the general solution is

$$c_1x^{\frac{1}{2}}+c_2\frac{1}{x}.$$

Before going on, look again at the definition of δ:

$$\delta y=xy'.$$

Notice in particular that if $y=x^r$, we get

$$\delta x^r=x\left(rx^{r-1}\right)=rx^r.$$

And look again at equation (8) :

$$\delta^2 x^r = r^2 x^r$$

We could keep going and get

$$\delta^n x^r = r^n x^r.$$

(This would be useful for higher order equations.) Compare this with D and the function e^{rx} :

$$\begin{aligned} De^{rx} &= re^{rx} \\ D^2 e^{rx} &= r^2 e^{rx}, \end{aligned}$$

and so forth. This is why the δ operator is useful. It acts on the function x^r in the same way as D does on e^{rx}. For Euler equations, at least, δ is a very convenient operator.

The complications that can occur in using it for Euler equations are the same as for the constant coefficient case. We can have repeated roots, or we can have complex roots.

If r is complex, we have to consider what is meant by x^r. We use the definition of $x^{r,}$ plus Euler's formula. Suppose that $r = \lambda + i\mu$. Then

$$\begin{aligned} x^r &= e^{r \log x} = e^{(\lambda + i\mu) \log x} = e^{\lambda \log x} \left(\sin\left(\mu \log x\right) + i \cos\left(\mu \log x\right)\right) \\ &= x^\lambda \left(\sin\left(\mu \log x\right) + i \cos\left(\mu \log x\right)\right) \end{aligned}$$

Two real solutions are then

$$y_1 = x^\lambda \sin\left(\mu \log x\right), \quad y_2 = x^\lambda \cos\left(\mu \log x\right).$$

If the only use of the delta method was to solve Euler equations, it would not be all that helpful, because we already know how to solve these. But we can also solve equations where series are needed. Here is an important example:

Bessel's equation of order $\frac{1}{2}$ (pg. 299 in the text):

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{4} \right) y = 0.$$

The term x^2y means it is not an Euler equation.

We use (7) to write this as

$$\left(\delta\left(\delta-1\right)+\delta-\frac{1}{4}\right)y=-x^2y,$$

putting all terms with x on the right. This is then

$$\left(\delta^2-\frac{1}{4}\right)y=-x^2y.$$

The indicial equation is found on the left:

$$r^2-\frac{1}{4}=0,$$

with solution $r_1=\frac{1}{2}, r_2=-\frac{1}{2}$. But for the moment, keep the general exponent r.

$$y=\sum_{n=0}^{\infty}a_nx^{n+r}.$$

We found earlier that

$$\begin{aligned}\delta x^{n+r}&=\left(n+r\right)x^{n+r}\\ \delta^2x^{n+r}&=\left(n+r\right)^2x^{n+r}.\end{aligned}$$

Our equation is

$$\left(\delta^2-\frac{1}{4}\right)y=-x^2y$$

and so we get

$$\sum_{n=0}^{\infty}\left(\left(n+r\right)^2-\frac{1}{4}\right)a_nx^{n+r}=-\sum_{n=0}^{\infty}a_nx^{n+r}=-\sum_{n=2}^{\infty}a_{n-2}x^{n+r}$$

$$x^r\left(r^2-\frac{1}{4}\right)a_0+x^r\left(\left(r+1\right)^2-\frac{1}{4}\right)a_1+x^r\sum_{n=2}^{\infty}\left\{\left(\left(n+r\right)^2-\frac{1}{4}\right)a_n+a_{n-2}\right\}x^n=0.$$

Setting $r=\frac{1}{2}$ and $a_1=0$ eliminates the first two terms. Then we get the recurrence relation

$$a_n=-\frac{a_{n-2}}{\left(n+\frac{1}{2}\right)^2-\frac{1}{4}}=-\frac{a_{n-2}}{n\left(n+1\right)},n\geq 2$$

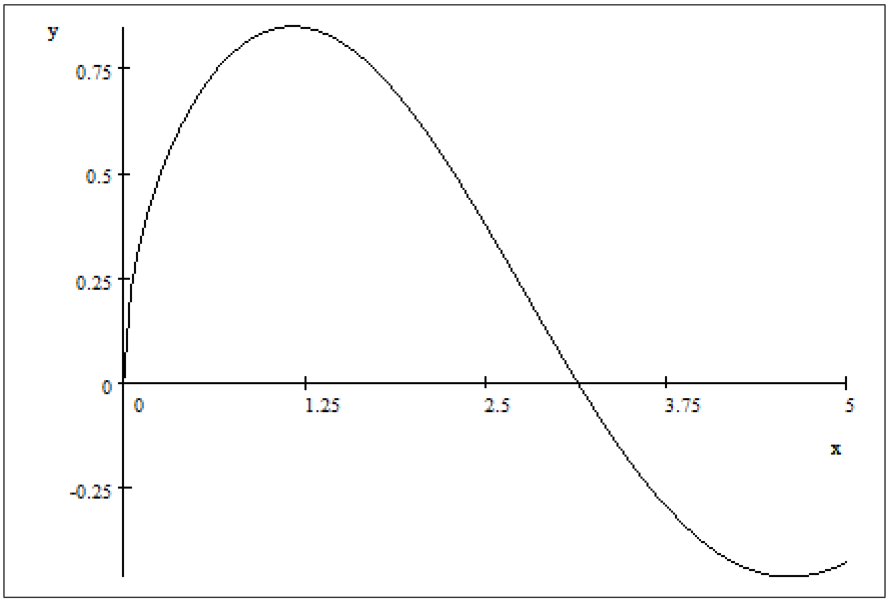
and this gives us one solution.

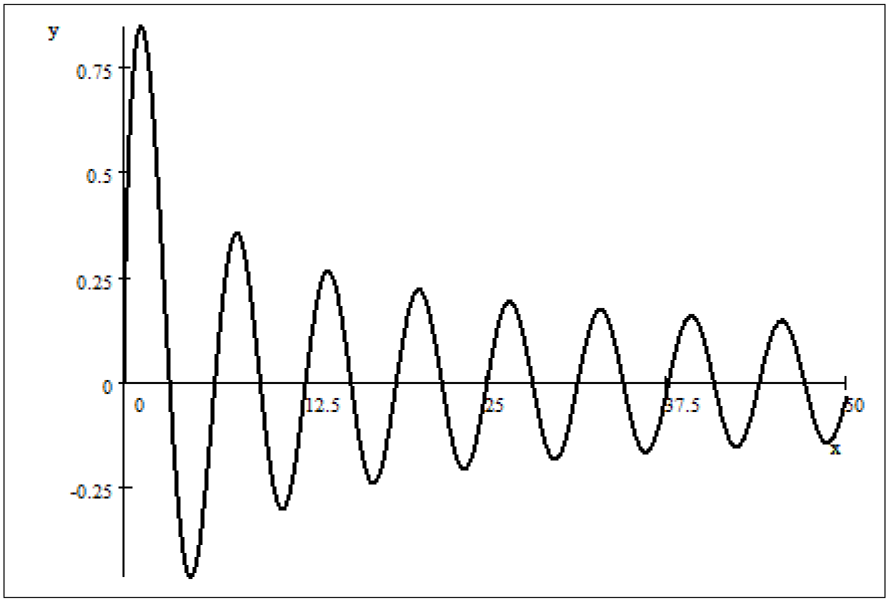
It is interesting to evaluate the first few terms of this series. With $a_0 = 1$, we get

$$
\begin{aligned}
a_2 &= -\frac{1}{3!} \\
a_4 &= -\frac{1}{(5)(4)} a_2 = \frac{1}{5!}
\end{aligned}
$$

So, we see that our solution is

$$
\begin{aligned}
& x^{\frac{1}{2}} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots \right) \\
&= x^{-\frac{1}{2}} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \right) = x^{-\frac{1}{2}} \sin x.
\end{aligned}
$$





A certain multiple of his function is called the "Bessel function of order $\frac{1}{2}$ of the first kind..(See text.) It is denoted by $J_{\frac{1}{2}}(x)$.

For the second solution, $r = -\frac{1}{2}$, and

$$a_n = -\frac{a_{n-2}}{\left(n-\frac{1}{2}\right)^2 - \frac{1}{4}} = -\frac{a_{n-2}}{n(n-1)}, n \geq 2$$

$$x^{-\frac{1}{2}}\left(1 - \frac{1}{2}x^2 + \frac{1}{4!}x^3 + \cdots\right) = x^{-\frac{1}{2}} \cos x$$

This is called a "Bessel function of order $\frac{1}{2}$ of the second kind", and denoted by $Y_{\frac{1}{2}}(x)$.

2 Homework, Due Wednesday, March 24.

This is the last assignment before the second exam, which is on Friday, March 26.

In the 9th edition: pg. 276, # 5, 14, 20, page 283, # 13. Do by the "delta" method from the notes.

In the 8th edition: pg. 278, # 5, .14 (please do these first, to help the grader by having problems in the same order.

pg. 271, # 4.

pg. 285, # 13. Do by the “delta” method from the notes.