Math 1280
Notes 13

In this last set of notes I will try to tie up some loose ends.

1 A complete Fourier series solution

First here is an example of the full solution of a pde by Fourier series.

Consider the wave equation

$$u_{tt} = a^2 u_{xx} \tag{1}$$

with boundary conditions

$$u(0,t) = u(1,t) = 0 \tag{2}$$

and initial conditions

$$u(x,0) = 1,\ \ 0 \le x \le 1 \tag{3}$$
$$u_t(x,0) = 0,\ \ 0 \le x \le 1. \tag{4}$$

We use separation of variables, writing

$$u(x,t) = X(x)\,T(t)\,,$$

from which we get

$$\frac{a^2 X''}{X} = \frac{T''}{T} = -\mu^2,$$

by the argument we have used before. This gives

$$X'' + \left(\frac{\mu}{a}\right) X = 0 \tag{5}$$
$$T'' + \mu T = 0. \tag{6}$$

From the first equation and the boundary conditions we get

$$\begin{aligned} X_n &= \sin n\pi x \\ T_n &= c_1 \cos(n\pi a t) + c_2 \sin n\pi a t. \end{aligned}$$

The 2nd initial condition, (4) requires that

$$T_n'(0) = 0,$$

from which we derive that $c_2 = 0$, and we can set $c_1 = 1$. This gives us the Fourier sine series solution

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin n\pi x \cos n\pi a t.$$

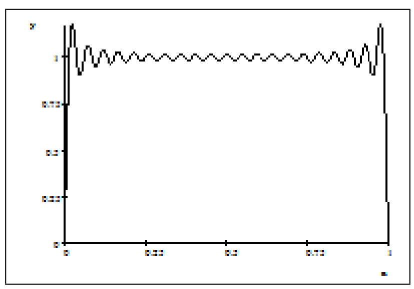
Now we will assume that $a = 1$, as in the homework problems. To satisfy the initial condition (3), we need

$$a_n = 2\int_0^1 \sin n\pi x \, dx = 2\left\{-\frac{1}{n\pi}\cos n\pi x\right\}|_0^1 = -\frac{2}{n\pi}\left((-1)^n - 1\right) = \begin{cases} 0 \text{ if } n \text{ is even} \\ \frac{4}{n\pi} \text{ if } n \text{ is odd} \end{cases}.$$

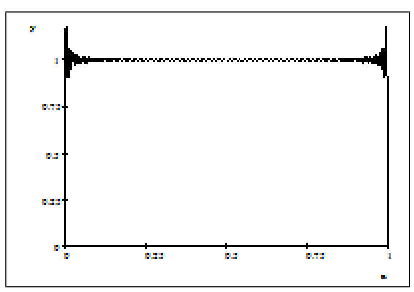
Hence, setting $m = 2n+1$ to give only the odd terms,

$$u(x,t) = \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi} \sin(2m+1)\pi x \ \cos(2m+1)\pi t.$$

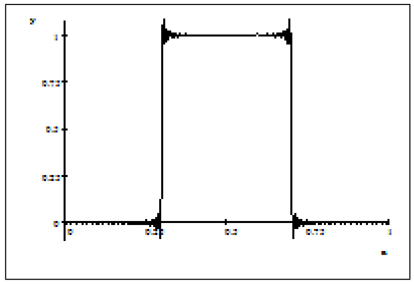
Now we wish to see how this solution behaves. We start by testing the initial condition Below I plot the sum of the first 20 terms.



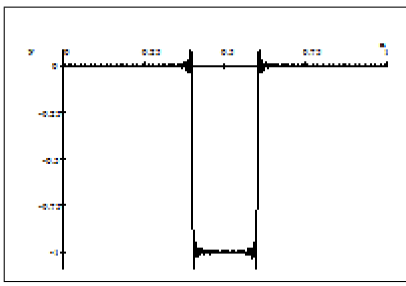
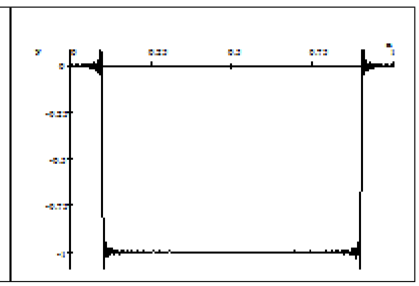
This looks correct. Now I'll plot the first 100 terms.



Next I'll plot the first 100 terms when $t = 0.3$.



In class I showed some more, indicating that the region where $u = 0$ spreads from the two boundary points, $x = 0$ and $x = 1$ into the center, so that $u(x, 0.5) = 0$. In fact, we know that $\cos \frac{2m+1}{2} = 0$ for any m. I'll give two more plots here:

2 Sturm-Liouville problems

These are discussed in section 11.2. They are generalizations of the two point boundary value problems studied earlier.

A Sturm-Liouville ("**lyoo** vil") consist of a differential equation of the form

$$p(x)\, y'' + p'(x)\, y' - q(x)\, y + \lambda r(x)\, y = 0 \tag{7}$$

with one of two kinds of boundary conditions, either "separated"

$$\begin{aligned} ay(0) + by'(0) &= 0 \\ cy(x) + dy'(1) &= 0, \end{aligned}$$

or "periodic":

$$\begin{aligned} y(0) &= y(1) \\ y'(0) &= y'(1)\,. \end{aligned}$$

We require that $r(x) > 0$ for $0 \leq x \leq 1$. We also require that $p(x) \neq 0$ for $0 < x < 1$. If $p(0) = 0$ or $p(1) = 0$, then the problem is called "singular".

The minus sign in the differential equation is standard, but won't affect what we do. You can always get a minus sign there by changing what you call $q(x)$ if necessary.

The differential equation is usually written as

$$(p(x)\,y')' - q(x)\,y + \lambda r(x)\,y = 0. \tag{8}$$

Note that by using the product rule in the first term of (8) you get (7).

We will only consider two examples:

$$y'' + \lambda y = 0,$$

which is already familiar, and

$$(xy')' + \lambda^2 xy = 0.$$

If we use the product rule and multiply by x, we get

$$x^2 y'' + xy' + \lambda^2 x^2 y = 0. \tag{9a}$$

Referring to the notes on Bessel's equation, we see that this is Bessel's equation of order zero. In the notes on Laplace's equation in polar coordinates, we ran into this equation, except that we wrote it as

$$r^2 R'' + rR' + \lambda^2 r^2 R = 0.$$

This is why we are discussing it here.

Definition: *An eigenvalue for a Sturm-Liouville problem is a number* λ such that there is a nonzero solution $\phi(x)$ to the boundary value problem. The solution ϕ is called an eigenfunction associated with λ. Notice that if ϕ is an eigenfunction, then so is $c\phi$, for any $c \neq 0$.

Here are some important facts about Sturm-Liouville problems:

1. They have an infinite sequence of eigenvalues $\lambda_1 < \lambda_2 < \cdots$, with associated

2. If ϕ_i and ϕ_j are eigenfunctions associated with distinct eigenvalues λ_i and λ_j $(i \neq j)$, then

$$\int_0^1 r(x)\,\phi_i(x)\,\phi_j(x)\,dx = 0.$$

This is an orthogonality condition.

Notice that for the equation

$$y'' + \lambda y = 0,$$

the eigenfunction are sines or cosines, or both, depending on the boundary conditions. In this case we know that the eigenfunctions are orthogonal, because we can actually do the integral. For Bessel's equation the eigenfunctions are $J_0(\lambda_i x)$, where the λ_i are the zeros of J_0. That is, $J_0(\lambda_i) = 0$. In this singular case, we know that $J_0(0) = 1$, so there is no way to satisfy the boundary condition $y(0) = 0$. Because the equation is singular at 0, the boundary condition there is changed. We look for solutions to $(9a)$ such that

$$\begin{aligned} y \text{ is bounded on } (0,1) \\ y(1) = 0. \end{aligned}$$

The second solution of Bessel's equation involves $\ln x$, and this is unbounded on $(0,1)$, because $\lim_{x\to 0^+} \ln x = -\infty$. Hence any eigenfunction is a multiple of $J_0(\lambda_i x)$.

We wish to prove the orthogonality We will do this for Bessel's equation. This is problem 14 on page 306. I am not assigning it, but doing the main steps for you.

In this case, $r(x) = x$. We wish to show that

$$\int_0^1 x J_0(\lambda_i x)\, J_0(\lambda_j x)\, dx = 0.$$

To do this, we write down the differential equations satisfied by $J_0(\lambda_i x)$ and $J_0(\lambda_j x)$. These are simpler to look at if we let $y_1(x) = J_0(\lambda_1 x)$ and $y_2 = J_0(\lambda_2 x)$ and work with y_1 and y_2. Any other λ_i and λ_j can be treated the same way, as long as $\lambda_i \neq \lambda_j$.

We have

$$\begin{aligned} (xy_1')' + \lambda_1^2 x y_1 = 0 \\ (xy_2')' + \lambda_2^2 x y_2 = 0. \end{aligned}$$

Multiply the first equation by y_1, the second by y_2 and subtract. We get

$$y_2\left(xy_1'\right)' - y_1\left(xy_2'\right)' + \left(\lambda_1^2 - \lambda_2^2\right)xy_1y_2 = 0.$$

Now integrate this from $x = 0$ to $x = 1$. This gives

$$\int_0^1 \left[y_2\left(xy_1'\right)' - y_1\left(xy_2'\right)'\right]dx = \left(\lambda_2^2 - \lambda_1^2\right)\int_0^1 xy_1y_2\ dx = 0. \tag{10}$$

The trick is to use integration by parts. Look at this carefully. We have

$$\int_0^1 y_2(x)\left(xy_1'(x)\right)' dx = y_2(x)\left(xy_1'(x)\right)|_0^1 - \int_0^1 xy_1'(x)\,y_2'(x)\,dx.$$

Doing the same with the second term in the integral on the left of (10), we find that

$$\int_0^1 xy_1y_2dx = y_2\left(xy_1'\right)|_0^1 - y_1\left(xy_2'\right)|_0^1 + \frac{1}{\lambda_2^2 - \lambda_1^2}\left(\int_0^1 xy_1'y_2'dx - \int_0^1 xy_1'y_2'dx\right). \tag{11}$$

But the second term is zero, because the integrals are equal So we only have to look at the terms which are evaluated at the boundary, $x = 0$ and $x = 1$.

For this problem we have the boundary conditions

$$y(0)\ \text{bounded},\ y(1) = 0.$$

Then y_1 and y_2 both satisfy these conditions. Look at the term

$$y_2(x)\left(xy_1'(x)\right)$$

evaluated at $x = 0$. Clearly this is zero, because of the term x. Then look at this term at $x = 1$. This time it is zero because $y_1(1) = 0$. In the same way, every term we get on the right of (11) is zero, and this proves that

$$\int_{01} xy_1y_2dx = 0.$$

This proves the orthogonality of $J_0(\lambda_1 x)$ and $J_0(\lambda_2 x)$ with respect to the weighting function $r(x) = x$.