1 Laplace’s equation in polar coordinates

Recall that Laplace’s equation is
\[ u_{xx} + u_{yy} = 0. \]

This is usually considered in some region \( \Omega \) of the \( x, y \) plane. For example, we considered it in the previous notes when \( \Omega \) was the rectangle
\[ \Omega = \{(x, y) \mid 0 \leq x \leq L, 0 \leq y \leq 1\}. \]

We could find solutions, because we can write \( u(x, y) = X(x)Y(y) \), and the functions \( X(x) \) and \( Y(y) \) each are defined over a fixed interval. The range of \( y \) does not depend on \( x \), and the range of \( x \) does not depend on \( y \).

Now we want to consider Laplace’s equation in the disk
\[ \Omega = \{(x, y) \mid 0 \leq x^2 + y^2 \leq 1\}. \]

But this seems to be difficult, because we can’t expect to write
\[ u(x, y) = X(x)Y(y), \]
because in \( \Omega \), the range of values of \( y \) depends on \( x \). That is, we could write \( \Omega \) as
\[ \Omega = \{(x, y) \mid -1 \leq x \leq 1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}\}. \]

For this region, it is reasonable to use polar coordinates. If \( x = r \cos \theta \) and \( y = r \sin \theta \), then
\[ \Omega = \{(r \cos \theta, r \sin \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}. \]

Note that the range of \( \theta \) does not depend on \( r \) and the range of \( r \) does not depend on \( \theta \).

Now it is reasonable to write \( u \) as a function of \( r, \theta \). Saying that differently, we define a function \( v(r, \theta) \) by the equation
\[ v(r, \theta) = u(r \cos \theta, r \sin \theta). \]
We then want to find a differential equation satisfied by \( v \). We can do this by differentiating using the chain rule. For example,

\[
v_r (r, \theta) = \frac{\partial u}{\partial x} (r \cos \theta, r \sin \theta) \frac{\partial (r \cos \theta)}{\partial r} + u_y (r \cos \theta, r \sin \theta) \frac{\partial (r \sin \theta)}{\partial r}
\]

\[= u_x (r \cos \theta, r \sin \theta) \cos \theta + u_y (r \cos \theta, r \sin \theta) \sin \theta.
\]

This is already complicated, and we expect to need to know \( v_{rr} \) and \( v_{\theta\theta} \). I will not give these complications here, but will put a derivation of the equations satisfied by \( v \) on the website. It was also passed out in class. The result is that \( v \) satisfies the equations

\[v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 0.
\] (1)

But often, as in the notes on this, the terminology \( v (r, \theta) \) is not introduced, and \( u \) is considered to be both a function of \( x \) and \( y \) and a function of \( r \) and \( \theta \). Thus, the following formula is derived in the notes on the Laplacian.

\[u_{xx} + u_{yy} = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}.
\]

This is sloppy notation, and a separate function \( v \) should be considered, but we will go along with standard terminology and use \( u \) for both functions.

If we consider \( u \) as a function of \( r \) and \( \theta \), then boundary conditions for (1) are easily given:

\[u (1, \theta) = f (\theta)
\]

for some function \( f (\theta) \).

Note that the correspondence between \((x, y)\) and \((r, \theta)\) is not 1:1, because the value of \( \theta \) is not uniquely determined by \((x, y)\). Thus, for this to make sense, we must require that

\[u (r, \theta) = u (r, \theta + 2\pi)
\]

for all \( \theta \). This, in turn, leads to the requirement that

\[f (\theta) = f (\theta + 2\pi).
\]

It makes sense to try to solve (1) by separation of \( r \) and \( \theta \). Thus, we assume that \( v (r, \theta) = R (r) \Theta (\theta) \). Then,

\[v_r = R' (r) \Theta (\theta), v_\theta = R (r) \Theta' (\theta),
\]
and differentiating again, and dividing by \( R (r) \Theta (\theta) \), we get
\[
\frac{R''}{R} + \frac{2 R'}{r R} = -\frac{1}{r^2} \frac{\Theta''}{\Theta} (\theta).
\]

Multiplying by \( r^2 \):
\[
\frac{r^2 R''}{R} + 2r \frac{R'}{R} = -\frac{\Theta''}{\Theta} (\theta).
\]

A function of \( r \) equals a function of \( \theta \), and so both must be constant.

Recall that we want \( \Theta \) to be periodic. This requires an equation with sines and cosines as solutions. Hence, we assume that
\[
\frac{r^2 R''}{R} + 2r \frac{R'}{R} = -\frac{\Theta''}{\Theta} (\theta) = \mu^2.
\]
so that
\[
\Theta'' + \mu^2 \Theta = 0,
\]
and
\[
\Theta (\theta) = c_1 \cos \mu \theta + c_2 \sin \mu \theta.
\]
Setting \( \Theta (0) = 0 \) gives \( c_1 = 0 \). To get \( \Theta (2\pi) = 0 \), we require \( \mu = n \) for some integer \( n \).

For the remaining of the analysis, see the last three pages of section 10.8. I will only point out that the equation for \( R \) becomes
\[
r^2 R'' + r R' - \mu^2 R = 0.
\]
This is an Euler equation, and so can be written using the \( \delta \) operator, as
\[
(\delta^2 - \mu^2) R = 0.
\]
We usually express the indicial equation in terms of a variable “\( r \)”, but we already have an \( r \) here, so we will write the indicial equation as
\[
\lambda^2 - \mu^2 = 0,
\]
from which we get \( \lambda = \pm \mu \), and the solutions are
\[
c_1 r^\mu + c_2 r^{-\mu}.
\]
We want \( u (x, y) \) to be defined at \( x = 0, y = 0 \), so we require that \( c_2 = 0 \). Also recall that \( \mu = n \).
2 Wave equation in a disk

In dimensions higher than 1, it is common to use a notation for Laplace’s equation which does not reflect a particular geometry, such as a rectangle or a disk. The symbol $\Delta$ is used for the operator

$$\Delta u = u_{xx} + u_{yy}.$$ 

If we are using polar coordinates, then

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta}.$$ 

We can then write the wave equation in two dimensions as

$$u_{tt} = \Delta u,$$ \hspace{1cm} (2)

no matter what coordinates we use to solve the equation. Once again, we will assume that this holds in

$$\Omega = \{(x, y) \mid 0 \leq x^2 + y^2 \leq 1\} = \{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}.$$ 

In order to be sure that a solution exists, and there is only one solution, we need some combination of initial conditions and boundary conditions. To get an idea of what these should be, recall the one-dimensional case, which we discussed in class and is covered in section 10.7 of the text. There we have

$$u_t = u_{xx}$$

and we considered the region $0 \leq x \leq 1$, $t > 0$. It was shown that to get a unique solution you want some boundary conditions, such as

$$u(0, t) = u(1, t) = 0,$$

and two initial conditions,

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x),$$

for $0 \leq x \leq 1$. The boundary conditions $u = 0$ at $x = 0$ and $x = 1$ correspond to a vibrating string which is fixed at $u = 0$ at each end.

The two dimensional case can represent a vibrating membrane, such as a drum head, and if we require that $u = 0$ at the boundary, we are simply saying that the
membrane is fastened down around the circumference of the drum. By using polar coordinates we are assuming that the drum is circular, which is certainly true of most drums.

As we saw earlier, the Laplacian operator can be expressed in polar coordinates as

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}. $$

So, the wave equation becomes

$$u_{tt} = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}. $$

This involves three independent variables, $t$, $r$, and $\theta$, and is difficult to solve. So we will simplify it by assuming that $u$ doesn’t depend on $\theta$. In this case, we get

$$u_{tt} = u_{rr} + \frac{1}{r} u_r. $$ (3)

Then we use the separation of variables technique, setting

$$u(r,t) = R(r) T(t).$$

Differentiating, and dividing by $RT$, gives the equation

$$\frac{T''(t)}{T(t)} = \frac{R''(r)}{R(r)} + \frac{1}{r} \frac{R'(r)}{R(r)},$$

and since the left side depends on $t$ and the right side on $r$, and $r$ and $t$ are independent of each other, each side must be constant. Following the text, we write

$$\frac{T''(t)}{T(t)} = \frac{R''(r)}{R(r)} + \frac{1}{r} \frac{R'(r)}{R(r)} = -\lambda^2,$$

and we have to determine allowable values of $\lambda$.

These will come from the $R$ equation, but we note that for each $\lambda$, the $T$ equation is

$$T'' + \lambda^2 T = 0,$$

with solution

$$a \cos \lambda t + b \sin \lambda t.$$

The $R$ equation is

$$r^2 R'' + r R' + \lambda^2 r^2 R = 0.$$
Here $\lambda$ is yet to be determined.

Supposing for the moment that we know $\lambda$, we can follow the text and introduce a new variable $\xi = \lambda r$. The equation becomes

$$\xi^2 \frac{d^2 R}{d\xi^2} + \xi \frac{dR}{d\xi} + \xi^2 R = 0. \quad (4)$$

this is Bessel’s equation of order zero. The only solutions which are defined at $\xi = 0$ are multiples of $J_0 (\xi) = J_0 (\lambda r)$.

To determine $\lambda$, we use the boundary conditions. The condition that $u = 0$ on the boundary becomes

$$u (1, t) = 0$$

for all $t > 0$.

Recall from notes 10 (as revised), that Bessel’s equation of order zero has two linearly independent solutions. One is

$$J_0 (\xi) = 1 - \frac{1}{4} \xi^2 + \frac{1}{64} \xi^4 - \ldots$$

and the other is

$$J_0 (\xi) \ln \xi + \sum_{m=1}^{\infty} b_m \xi^{2m}$$

for certain coefficients $b_m$. Since $\lim_{\xi \to 0^+} \ln \xi = -\infty$, the only solutions which are bounded in a bounded region containing 0 are multiples of $J_0$. Hence, we take $R (r) = J_0 (\lambda r)$. We stated earlier that $J_0$ has zeros at a sequence of numbers $\lambda_1, \lambda_2, \ldots$. Hence these are the eigenvalues for the problem

$$r^2 R'' + r R' + \lambda^2 r^2 R = 0 \quad (5)$$

$$R (0) \text{ finite, } R (1) = 0$$

(This is a different kind of boundary value problem from before. Since $J_0 (0) = 1$, we cannot find a solution with $R (0) = 0$. And physically, we don’t expect that. We expect that this drum is vibrating in the middle. But we do expect that the vibrations are of finite amplitude. Fortunately, the calculations give us exactly the kind of solution we want. (Otherwise, we would discard the equations as being a bad model of a vibrating drum, even one where the drum is only being hit in the center.))

We now recall, with a little more detail, some information about $J_0$. If we divide (4) by $\xi^2$, we get

$$R'' + \frac{1}{\xi} R' + R = 0.$$
It is natural to think that the term \( \frac{1}{\xi} R' \) becomes less important as \( \xi \to \infty \), and in a sense this is true. If that term weren’t there, we would have \( R'' + R = 0 \), with sine and cosine as solutions. Both sine and cosine are zero at an infinite number of points, separated by \( \pi \). It can be shown that \( J_0 \) also has an infinite sequence of zeros, at numbers \( \lambda_1, \lambda_2, \lambda_3, \ldots \), and in fact, \( \lim_{j \to \infty} (\lambda_{j+1} - \lambda_j) = \pi \). Thus, \( R(1) = 0 \) if, in equation (5), \( \lambda = \lambda_j \) for some \( j \). We therefore get an infinite sequence of solutions to (9), of the form

\[
J_0 (\lambda_n r) \left( a_n \cos \lambda_n t + b_n \sin \lambda_n t \right).
\]

The sum of solutions is still a solution, so we look for a solution in the form

\[
\sum_{n=0}^{\infty} J_0 (\lambda_n r) \left( a_n \cos \lambda_n t + b_n \sin \lambda_n t \right).
\]

Now we have to satisfy the initial conditions, which were \( u(r,0) = f(r), u_t(r,0) = g(r) \).

Following the book, we will set \( g(r) = 0 \), and this results in \( b_n = 0 \). We therefore have to satisfy the equation

\[
\sum_{n=0}^{\infty} a_n J_0 (\lambda_n r) = f(r).
\]

This is like a Fourier series, but with \( J_0 (\lambda_n r) \) instead of sines or cosines. It turns out that we can find the \( a_n \) by a method similar to that used for Fourier series. This is possible because of the following orthogonality relation for \( J_0 \):

\[
\int_0^1 r J_0 (\lambda_n r) J_0 (\lambda_m r) \, dr = 0
\]

if \( n \neq m \). We will go over this in class.

### 3 Homework

pg. 632, # 1 (a,b,c,e)

pg. 675, # 8. But find the expansions for each of the sets of eigenfunctions for problems 1 and 2. Plot the sum of the first 10 terms in each case. Each graph should be an approximation of the function \( f(x) \) given in # 8. on \( 0 \leq x \leq 1 \), but plot on \( 0 \leq x \leq 2 \).

pg. 706, # 4