Math 1280
Spring, 2010

Notes on phase planes

1 Some basic facts about solution trajectories

Consider a general system of two autonomous equations in two unknowns:

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned} \tag{1}$$

If $(x(t), y(t))$ is a solution to (1), then as t varies, the point $(x(t), y(t))$ moves along a curve in the x, y plane. This curve is called a trajectory, or orbit, or phase curve, for the system (1).

We first want to recall an existence and uniqueness theorem for the system (1).

Theorem 1 *Suppose that the partial derivatives* $\frac{\partial f(x,y)}{\partial x}, \frac{\partial f(x,y)}{\partial y}, \frac{\partial g(x,y)}{\partial x}$, *and* $\frac{\partial g(x,y)}{\partial y}$ *are continuous. Then for any point* (x_0, y_0) *in the* x, y *plane, there is a unique solution of* (1) *satisfying the initial conditions*

$$\begin{aligned} x(0) &= x_0 \\ y(0) &= y_0. \end{aligned}$$

The whole concept of the phase plane depends on this theorem. If we specify a point in the plane, we want only one phase curve to pass through this point. Otherwise there is too much confusion and we can't draw a meaningful picture.

As an example, consider

$$\begin{aligned} x' &= 3x^{\frac{2}{3}} \\ y' &= 3y^{\frac{2}{3}}. \end{aligned}$$

Note that this system does not satisfy the hypotheses of the theorem at $(x_0, y_0) = (0,0)$. What is the solution such that $x(0) = 0$ and $y(0) = 0$? There are at least three of them:

$$\begin{aligned} (x(t), y(t)) &= \left(t^3, 0\right) \\ (x(t), y(t)) &= \left(0, t^3\right) \\ (x(t), y(t)) &= \left(t^3, t^3\right). \end{aligned}$$

You can easily see that each of these is a solution to the system. But they have three completely different phase curves: The x- axis, the y-axis, and the line $y = x$.

Suppose, for example, that a particle moved so as to satisfy this system of equations, that is, its position at time t was given by $(x(t), y(t))$, and this particle started at $(0, 0)$. Where would it go?

So we will always assume that our system satisfies the hypotheses of this theorem. **We will not bother to say this every time we consider such a system.**

Definition 2 *A point* (x_0, y_0) *is called an equilibrium point for the system* (1) *if* $f(x_0, y_0) = g(x_0, y_0) = 0$.

Observe that if (x_0, y_0) is an equilibrium point, then the constant functions $x(t) = x_0$, $y(t) = y_0$ solve the system with the initial condition $x(0) = x_0, y(0) = y_0$. And with our assumption that the partial derivatives of f and g are continuous, this constant solution is the only one that starts at this equilibrium point. In fact, it is the only solution that "passes through" this point at any value of t.

To see this, suppose that $(x(t), y(t))$ is a solution such that for some value of t, say t_1, $x(t_1) = x_0$ and $y(t_1) = y_0$. Consider the new functions

$$u(t) = x(t + t_1), v(t) = y(t + t_1).$$

Then $u(0) = x(t_1) = x_0$, $v(t) = y(t_1) = y_0$. Also, I claim that $(u(t), v(t))$ is a solution of the system (1). We check this by differentiating and using the definitions of u and v :

$$\begin{aligned} u'(t) &= x'(t + t_1) = f(x(t + t_1), y(t + t_1)) = f(u(t), v(t)) \\ v'(t) &= y'(t + t_1) = g(x(t + t_1), y(t + t_1)) = g(u(t), v(t)). \end{aligned}$$

(Note that this only works because f and g do not depend explicitly on t. The system is what we call "autonomous".)

Since the only solution of (1) such that $x(0) = x_0, y(0) = y_0$ is the constant solution, u and v must be this solution, and so as well, $x(t) = x_0, y(t) = y_0$ for all t .

A similar proof shows that no two trajectories can pass through the same point, whether or not it is an equilibrium point.

2 Classification of linear systems of two equations with constant coefficients

(This material is also in section 9.1 of the text, but with different notation. You can read either the text or these notes or both, as you wish.)

Consider a system

$$x' = ax + by$$
$$y' = cx + dy$$

where a, b, c, d are constants. As we know, this can be written in vector form, as

$$\mathbf{x}' = A\mathbf{x}, \tag{2}$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

One solution to (2) is clearly the constant solution

$$\mathbf{x}(t) = \mathbf{0}.$$

In other words, $(0,0)$ is an equilibrium point for this system.

To find other solutions of (2) we first find the eigenvectors and eigenvalues. Suppose that the eigenvalues are r_1 and r_2. The phase plane can be drawn once we know these and know the eigenvectors. There are a number of cases, as follows (the subdivisions correspond to those in the text, except that I will break them down into more subdivisions) :

1. real eigenvalues which are of the same sign, but are not equal to each other ($r_1 \neq r_2$, but $r_1 r_2 > 0$.)
   (a) $r_1 > 0, r_2 > 0$

    (b) $r_1 < 0, r_2 < 0$

2. real eigenvalues of opposite signs. We can assume $r_1 > 0, r_2 < 0$.
3. equal nonzero eigenvalues
    (a) two linearly independent eigenvectors
        i. $r_1 = r_2 > 0$
        ii. $r_1 = r_2 < 0$
    (b) only one linearly independent eigenvector
        i. $r_1 = r_2 > 0$
        ii. $r_1 = r_2 < 0$
4. complex eigenvalues $r_1 = \lambda + i\mu, r_2 = \lambda - i\mu$, where λ and μ are real and $\lambda \neq 0$, $\mu \neq 0$
    (a) $\lambda > 0$
    (b) $\lambda < 0$
5. pure imaginary eigenvalues $r_1 = \mu i$, $r_2 = -\mu i$, μ real and nonzero.

Each of the major categories has a different looking phase diagram, and associated with these are the following names:

1. improper node
2. saddle point
3. proper node
4. spiral point
5. center

The various subcategories for cases $1, 3, 4$, can also be divided into two cases, according to whether the equilibrium point $\mathbf{0}$ is "stable" or "unstable". $\mathbf{0}$ is called "asymptotically stable" if all other solutions tend to $\mathbf{0}$ as $t \to \infty$.

Another possibility is that of the center, where the phase curves are circles or ellipses. These do not tend to $\mathbf{0}$, but they also do not tend to infinity. They remain bounded. In this case $\mathbf{0}$ is called a stable equilibrium point. Otherwise the equilibrium point $\mathbf{0}$ is called unstable.

$\mathbf{0}$ is also said to be stable if it is asymptotically stable.

To see which cases are stable and which are unstable, recall that in any case where there are two linearly independent eigenvectors (complex or real) , solutions of the system (2) can be written as

$$\mathbf{x}(t) = c_1 e^{r_1 t} \mathbf{v}_1 + c_2 e^{r_2 t} \mathbf{v}_2,$$

If r_1 and r_2 are real, then $\lim_{t \to \infty} \mathbf{x}(t) = \mathbf{0}$ for every solution if and only if $r_1 < 0$ and $r_2 < 0$.

The case where there is only one eigenvector is a bit more complicated, and will be discussed later. But here is the result of the analysis:

(A) $\mathbf{0}$ is asymptotically stable in cases **1b, 3a(ii), 3b(ii), 4(b)**
(B) $\mathbf{0}$ is stable in all of the cases above and also in case **5**
(C) $\mathbf{0}$ is unstable in all other cases.

Putting all this together, we can say for example that in the case 3a(i), $\mathbf{0}$ is an unstable proper node, while in 4(b), $\mathbf{0}$ is an asymptotically stable spiral point.

3 Homework: Due at the beginning of class on Wednesday, January 13.

section 9.1, # 2, 5 (answers in back, but explain why and draw a picture.) Also give the stability properties. Be sure to include arrows on your trajectories.