1 Proof of part of the stable manifold theorem

The stable manifold theorem was stated, somewhat informally, as Theorem 1 in Notes 9. Here we will prove only the following parts.

**Theorem 1** Consider a system

\[
\begin{align*}
    x' &= F(x, y) \\
    y' &= G(x, y)
\end{align*}
\]

where all partial derivatives of $F$ and $G$, of any order, are continuous on $\mathbb{R}^2$. Suppose that $(x_0, y_0)$ is an equilibrium point for (1) and let

\[
A = \begin{pmatrix}
    F_x(x_0, y_0) & F_y(x_0, y_0) \\
    G_x(x_0, y_0) & G_y(x_0, y_0)
\end{pmatrix}.
\]

Let $r_1$ and $r_2$ be the eigenvalues of $A$.

1. If $r_1$ and $r_2$ both have negative real parts, then $(0, 0)$ is an asymptotically stable equilibrium point for (1). (This was defined in definition 3 of notes 8.)

2. If the eigenvalues of $A$ are $r_1$ and $r_2$, with $r_1 < 0$ and $r_2 > 0$, then $(x_0, y_0)$ is a saddle point for (1). (This was defined in Notes 9.)

**Remark 2** The following definition explains the title of this section:

**Definition 3** If $(x_0, y_0)$ is a saddle point for the system (1), then the “stable manifold of (1) at $(x_0, y_0)$” is the union of $\{(x_0, y_0)\}$ and the two trajectories which tend to $(x_0, y_0)$ as $t \to \infty$. The “unstable manifold” is the union of $\{(x_0, y_0)\}$ and the two trajectories which tend to $(x_0, y_0)$ as $t \to -\infty$.

**Example 4** For the system $x' = -x$, $y' = y$, the stable manifold is the entire $x$-axis, including $(0, 0)$, and the unstable manifold is the entire $y$-axis. You should draw the phase plane to understand this.
2 Proof of Part 1 of theorem

Proof. In the proof of part (1) we use the concept of the “norm” of a matrix. There are several ways to define the norm of a matrix, but a convenient one here is the following:

Definition 5 Let $M = (m_{i,j})_{1 \leq i, j \leq n}$ be an $n \times n$ matrix. Then the norm of $M$ is defined to be

$$||M|| = \sum_{i,j=1}^{n} |m_{i,j}|.$$ 

We will also use the alternative norm in $\mathbb{R}^n$ which we discussed in section 2.1 of notes 6. There we let

$$|x| = \sum_{i=1}^{n} |x_i|$$

where

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n.$$ 

From these definitions it is easy to see that for any $x$ in $\mathbb{R}^n$,

$$||Mx|| \leq ||M|| \ |x|.$$ (2)

We will be using these definitions only in $\mathbb{R}^2$.

Lemma 6 Under the hypotheses of part (1) of the theorem, there are numbers $K > 0$ and $\sigma > 0$ such that for any $t > 0$,

$$||e^{At}|| \leq Ke^{-\sigma t}.$$ 

Proof. First suppose that $A$ is diagonalizable. Then there is a nonsingular matrix $B$ such that $D = B^{-1}AB$ is diagonal, where $D = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$. Then $A = BDB^{-1}$.

Note that $e^{Dt} = \begin{pmatrix} e^{rt} & 0 \\ 0 & e^{rt} \end{pmatrix}$ and so $||e^{Dt}|| \leq 2e^{-\sigma t}$ where $-\sigma = \max (\text{Re} r_1, \text{Re} r_2)$.

Hence,

$$e^{At} = e^{(BDB^{-1})t} = I + (BDB^{-1})t + \frac{1}{2}(BDB^{-1})(BDB^{-1})t^2 + \cdots$$

$$= B \left( I + Dt + \frac{1}{2}D^2t^2 + \cdots \right) B^{-1} = Be^{Dt}B^{-1}.$$
From the definition of norm of a matrix it follows easily that $\|e^{At}\| \leq \|B\| \|e^{Dt}\| \|B^{-1}\| \leq \|B\| \|B^{-1}\| 2e^{-\sigma t}$.

The case where $A$ is not diagonalizable is a little more complicated. Suppose that $A = \begin{pmatrix} r & 1 \\ 0 & r \end{pmatrix}$. In Notes 8, where we defined the exponential of a matrix, we saw that if $J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then

$$e^{Jt} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}.$$  

It follows similarly that if $r_1 = r_2 = -\rho$, with $\rho > 0$, then

$$e^{\begin{pmatrix} -\rho & 1 \\ 0 & -\rho \end{pmatrix}^t} = \begin{pmatrix} e^{-\rho t} & te^{-\rho t} \\ 0 & e^{-\rho t} \end{pmatrix}.$$  

From this it is easy to see that we cannot take the number $\sigma$ in the lemma to be $\rho$. Instead, suppose we choose $\sigma = \frac{1}{2}\rho$. (Any $\sigma$ in $(0, \rho)$ will work.) Then the term of most interest is the term $te^{-\rho t}$. Note that

$$\frac{te^{-\rho t}}{e^{-\sigma t}} = \frac{te^{-\rho t}}{e^{-\frac{1}{2}\rho t}} = te^{-\frac{1}{2}\rho t}.$$  

The maximum of this function is seen to be $\frac{2}{\rho}$. We can then set

$$K = 2 + \max_{t \geq 0} \frac{te^{-\rho t}}{e^{-\frac{1}{2}\rho t}} = 2 + \frac{2}{e\rho}.$$  

The general nondiagonalizable case is similar. We choose $B$ so that $B^{-1}AB = \begin{pmatrix} r & 1 \\ 0 & r \end{pmatrix}$ where $r < 0$ is the only eigenvalue of $A$. □

For simplicity we will now assume that $(x_0, y_0) = (0, 0)$. We have seen in discussing the linearization of a system that we can write (1) as

$$x' = Ax + g(x)$$  

where

$$\lim_{\|x\| \to 0} \frac{\|g(x)\|}{\|x\|} = 0. \quad (4)$$

Recall that the norms $\|x\|$ and $|x|$ are equivalent topologically. Proving convergence in one norm is equivalent to proving convergence in the other. So instead of (4) we can write

$$\lim_{|x| \to 0} \frac{|g(x)|}{|x|} = 0. \quad (5)$$

3
Of course, the perturbation term $g(x(t))$ is not known, since $x(t)$ is not known. But if we treat this term, $g(x(t))$, as a known function, we can apply the variation of parameters formula to (3). This is a very important technique in the rigorous treatment of nonlinear equations. Recalling that a fundamental solution of $y' = Ay$ is given by $e^{At}$, we see that if $u$ is a solution of (3), then

$$u(t) = e^{At}u(0) + \int_0^t e^{A(t-s)}g(u(s))\,ds \tag{6}$$

This is not a formula for the solution, since the unknown vector function $u$ appears on both sides of the equation. But it replaces the differential equation with an integral equation (a different integral equation from the one used in the fundamental existence and uniqueness theorem).

Hence, using (2) and Lemma 6,

$$|u(t)| \leq Ke^{-\sigma t}|u(0)| + K\int_0^t e^{-\sigma(t-s)}|g(u(s))|\,ds. \tag{7}$$

Now choose some $\varepsilon$ with $0 < \varepsilon < \sigma$. Then from (5) there is a $\delta > 0$ such that if $|x| \leq \delta$, then $|g(x)| \leq \frac{e|x|}{K}$. Suppose that $|u(0)| < \delta$. Then $|u(t)| < \delta$ on some maximal (largest possible) interval $[0, T)$. (It might be that $T = \infty$.) Then from (7) we see that on this interval

$$e^{\sigma t}|u(t)| \leq K|u(0)| + \varepsilon\int_0^t e^{\sigma s}|u(s)|\,ds. \tag{8}$$

At this point we are dealing only with numerical valued functions. The following Lemma is then used. It is a common tool in ode analysis.

**Lemma 7 (Gronwall’s Lemma)** Suppose that $w : [0, T) \to \mathbb{R}$ is continuous and satisfies the integral inequality

$$w(t) \leq a + b\int_0^t w(s)\,ds$$

for some $a$ and some $b > 0$. Then for all $t$ in $[0, T)$,

$$w(t) \leq ae^{bt}.$$

**Proof.** You will be asked to prove this result for homework. A hint will be given. ■

Apply this lemma to (8) with $w(t) = e^{\sigma t}|u(t)|$. We obtain that
\[ e^{|\varepsilon t|} u(t) \leq K |u(0)| e^{\varepsilon t}, \]

or

\[ |u(t)| \leq K |u(0)| e^{(\varepsilon-\sigma)t}. \]

We can assume that \( K \geq 1. \) But \( \varepsilon - \sigma < 0. \) Hence, if we restrict \( u(0) \) further, requiring that \( |u(0)| \leq \frac{\delta}{2K}, \) then \( |u(t)| \leq \frac{\delta}{2} \) for all \( t \) in \([0, T]\). Hence, if \( T \) is finite, then the solution remains with \( |u| < \delta \) for a little longer. Therefore \( T \) cannot be maximal unless \( T = \infty \), which proves stability. Furthermore, \( \lim_{t \to \infty} |u(t)| = 0, \) proving asymptotic stability. This completes the proof of part 1 of the theorem. ■

3 Outline of proof of part 2.

For part (2) of the theorem we will assume that \( r_1 = -1 \) and \( r_2 = +1, \) and also that the system has already been transformed to the form

\[
\begin{aligned}
x_1' &= -x_1 + g_1(x_1, x_2) \\
x_2' &= x_2 + g_2(x_1, x_2).
\end{aligned}
\]

This is done by replacing \( \mathbf{x} \) by \( M^{-1} \mathbf{x}, \) where \( M^{-1}AM = D, \) the diagonal matrix with \(-1\) and \( 1 \) on the diagonals. Further,

\[
\lim_{x_1^2 + x_2^2 \to 0} \frac{|g_i(x_1, x_2)|}{\sqrt{x_1^2 + x_2^2}} = 0.
\]

This implies that \( \frac{\partial g_i}{\partial x_j} (0, 0) = 0. \)

The technique now is to write this as an integral equation in the correct way, using the variation of parameters formula in such a way that only solutions which tend to zero as \( t \to \infty \) are included. Using indefinite integrals, the variation of parameters formula applied to each equation separately gives something of the form

\[
\begin{aligned}
x_1(t) &= ce^{-t} + \int e^{s-t} g_1(x_1(s), x_2(s)) \, ds. \\
x_2(t) &= de^{t} + \int e^{t-s} g_2(x_1(s), x_2(s)) \, ds
\end{aligned}
\]

(9) (10)

But we must choose appropriate limits in the two integrals.
For the first, since we have a term $e^{-t}$ in the integral, we can expect that if \[ \int e^s g_1 (x_1(s), x_2(s)) \, ds \] is not too large, then $x_1$ will tend to zero. But in the second integral we have $e^t$ in both terms, and some cancellation is necessary to get the limit to be zero. Consider the following system of integral equations.

\[
\begin{align*}
x_1(t) &= c e^{-t} + e^{-t} \int_0^t e^s g_1 (x_1(s), x_2(s)) \, ds \\
x_2(t) &= e^t \left( d + \int_0^t e^{-s} g_2 (x_1(s), x_2(s)) \, ds \right)
\end{align*}
\]

As $t \to \infty$ we can expect that the integral in the second line will converge, because of the term $e^{-s}$. But we need to know that the solution exists on $[0, \infty)$. Assuming this, to get $x_2(\infty) = 0$ we must choose $d = - \int_0^\infty e^{-s} g_2 (x_1(s), x_2(s)) \, ds$ in order to have a chance that the product of $e^t$ with the term in parentheses goes to zero as $t \to \infty$. Note that

\[- \int_0^\infty + \int_0^t = - \int_t^\infty.
\]

We therefore consider the following system of equations:

\[
\begin{align*}
x_1(t) &= c e^{-t} + e^{-t} \int_0^t e^s g_1 (x_1(s), x_2(s)) \, ds \\
x_2(t) &= - e^t \left( \int_t^\infty e^{-s} g_2 (x_1(s), x_2(s)) \, ds \right).
\end{align*}
\]

Our goal is to find a unique solution to the pair of integral equations (11) – (12), for each $c$ in some neighborhood of 0. We need to use the properties of $g$.

It is not immediately obvious that an $x_1$ and $x_2$ satisfying these equations will tend to zero at infinity. This will be part of the proof. But it is interesting to note the following L’hôpital’s rule calculation: If $h$ is continuous on interval $[0, \infty)$, and $\lim_{t \to \infty} h(t) = 0$ then

\[
\lim_{t \to \infty} e^{-t} \int_0^t e^s h(s) \, ds = \lim_{t \to \infty} \frac{\int_0^t e^s h(s) \, ds}{e^t} = 0,
\]

\[
= \lim_{t \to \infty} \frac{e^t h(t)}{e^t} = 0,
\]
and \( \int_t^\infty e^{-s}h(s)\,ds \) converges, with
\[
\lim_{t \to \infty} e^t \int_t^\infty e^{-s}h(s)\,ds = \lim_{t \to \infty} \frac{\int_t^\infty e^{-s}h(s)\,ds}{e^{-t}} = \lim_{t \to \infty} \frac{-e^{-t}h(t)}{-e^{-t}} = 0.
\]

(You should check that the assumptions of L’hôpital’s rule are satisfied.)

We now construct a solution to (11) – (12) by successive approximations, with starting functions
\[
\begin{align*}
x_1^0(t) &= ce^{-t} \\
x_2^0(t) &= 0
\end{align*}
\]
(13)

Assume that \( \int_t^\infty e^{-s}g_2(x^0(s))\,ds \) converges. We then set
\[
\begin{align*}
x_1^1(t) &= ce^{-t} + e^{-t} \int_0^t e^s g_1(x_1^0(s), x_2^0(s))\,ds \\
x_2^1(t) &= -e^t \int_t^\infty e^{-s}g_2(x_1^0(s), x_2^0(s))\,ds.
\end{align*}
\]

Once we show that \( x_1^1 \) and \( x_2^1 \) satisfy appropriate bounds, we can define \( x_1^2 \) and \( x_2^2 \) and so forth. In the following optional section, continued in the appendix, we show that the resulting sequence of functions converges to a solution of (9)-(10) which tends to \((0, 0)\) as \( t \to \infty \).

We can now give the formula of the local stable manifold as follows. For any sufficiently small \( c \) there is a unique solution \((x_1(t, c), x_2(t, c))\), the solution with \( x_1(0) = c \) and \( d = x_2(0) \) given by
\[
x_2(0, c) = d = -\int_0^\infty e^{-s}g_2(x_1(s, c), x_2(s, c))\,ds,
\]
(14)
which remains in \(|x| \leq \rho_0\) and tends to zero as \( t \to \infty \). In the optional section below inequality (16) implies that this solution tends to zero exponentially fast. Equation (14) is a formula for the local part of the stable manifold as in the statement of the theorem. It gives the part of the stable manifold near \((0, 0)\) as a smooth function \( d = h(c) \). Further away from \((0, 0)\) it may not be possible to express \( d \) as a function of \( c \), because the trajectories involved may not be the graphs of functions (being “multivalued”).

To obtain the unstable manifold we can reverse time in the system. That is, let \( p(t) = x(-t) \) Then \( p \) satisfies the system
\[
p' = -Ap - g(-p).
\]
The stable manifold for this system is the unstable manifold for the original system.

### 3.1 optional section, not done in class

In the following proof we will use the “sup norm” $|x|_{\infty} = \max \{|x_1|, |x_2|\}$, where $x = (x_1, x_2)$. We recall that $\frac{\partial g}{\partial x_j}(0,0) = 0$, and that the partial derivatives $\frac{\partial g}{\partial x_j}$ are continuous in $\mathbb{R}^2$. It follows that for any $\rho > 0$, $g = (g_1, g_2)$ must satisfy a Lipschitz condition of the form

$$|g(x_1) - g(x_2)|_{\infty} \leq L(\rho) |x_1 - x_2|_{\infty}$$

(15)

for all $x_1, x_2$ in the region $|x|_{\infty} \leq \rho$, and further, $\lim_{\rho \to 0} L(\rho) = 0$. In particular, this inequality holds with $x_2 = 0$ and so for any $x$ with $|x|_{\infty} \leq \rho$,

$$|g(x)|_{\infty} \leq L(\rho) |x|_{\infty}.$$

Now suppose that for some $\varepsilon \in (0,1)$ and some $\rho > 0$, $x(t) = (x_1(t), x_2(t))$ is a continuous vector valued function on $[0, \infty)$, and

$$|x(t)|_{\infty} \leq \rho e^{-(1-\varepsilon)t} \text{ for all } t \geq 0.$$  

(16)

Then

$$|g(x(t))|_{\infty} \leq L(\rho) \rho e^{-(1-\varepsilon)t}$$

for all $t \geq 0$. Recalling the definition of the norm of a vector, this inequality is also satisfied by each $g_i$ individually. We then have

$$\int_0^t e^{-(t-s)} |g_1(x_1(s), x_2(s))|_{\infty} ds \leq \int_0^t e^{-(t-s)} e^{-s+\varepsilon s} L(\rho) \rho ds$$

$$= \frac{\rho L(\rho)}{\varepsilon} e^{-t} (e^{\varepsilon t} - 1) \leq \frac{\rho L(\rho)}{\varepsilon} e^{-(1-\varepsilon)t},$$

(17)

and

$$\int_t^\infty e^{(t-s)} |g_2(x_1(s), x_2(s))|_{\infty} ds \leq \int_t^\infty e^{(t-s)} e^{-(1-\varepsilon)s} L(\rho) \rho ds = \frac{\rho L(\rho)}{2-\varepsilon} e^{-(1-\varepsilon)t}.$$  

(18)

**Proof.**

\[ \blacksquare \]
Pick \( \rho_0 > 0 \) such that
\[
L(\rho_0) < \frac{1}{2} \varepsilon,
\]
and assume that \( |c| \leq \frac{1}{2} \rho_0 \). With \( \mathbf{x} = (x_1, x_2) \), we have
\[
|\mathbf{x}^0(t)| \leq \rho_0 e^{-\varepsilon t} \leq \rho_0
\]
and so
\[
|\mathbf{g}(\mathbf{x}^0)| \leq \rho_0 L(\rho_0) e^{-\varepsilon t}.
\]
From (17), (18), and the assumptions \( |c| \leq \frac{1}{2} \rho_0 \) and \( L(\rho_0) \leq \frac{1}{2} \varepsilon \), we obtain
\[
|x_1^1(t)| \leq \frac{1}{2} \rho_0 e^{-t} + \frac{\rho_0 L(\rho_0)}{\varepsilon} e^{-(1-\varepsilon)t} \leq \rho_0 e^{-(1-\varepsilon)t}
\]
and so
\[
|x_2^1(t)| \leq \frac{\rho_0 L(\rho_0)}{2 - \varepsilon} e^{-(1-\varepsilon)t} \leq \rho_0 e^{-(1-\varepsilon)t},
\]
and so
\[
|\mathbf{x}^1(t)| \leq \rho_0 e^{-(1-\varepsilon)t}.
\]
Hence we can define \( \mathbf{x}^2, \mathbf{x}^3, \) etc. and all will satisfy this same bound. If \( \{\mathbf{x}^i\} \) converges uniformly on \([0, \infty)\) as \( i \to \infty \), then the limit functions satisfy the same bound on \([0, \infty)\). In particular, the limit function tends to \( 0 \) as \( t \to \infty \).

We have left to show the uniform convergence of \( \{\mathbf{x}^i\} \) on \([0, \infty)\).

The general inductive step is to set
\[
\begin{align*}
x_1^{j+1}(t) &= ce^{-t} + e^{-t} \int_0^t e^s g_1(x_1^j(s), x_2^j(s)) \, ds \\
x_2^{j+1}(t) &= -e^t \int_0^t e^{-s} g_2(x_1^j(s), x_2^j(s)) \, ds.
\end{align*}
\]
(21)

From this it follows as in the usual successive approximations procedure that \( \mathbf{x}^0 + \sum_{j=1}^\infty (\mathbf{x}^j - \mathbf{x}^{j-1}) \) converges uniformly on \([0, \infty)\) to a unique solution \( \mathbf{x} \) of (11) – (12), for all \( t \). This step is similar to the proof of convergence in the original existence and uniqueness theorem, and I will leave it to the appendix of this set of notes.

4 Periodic solutions and limit cycles in the plane

Again we are considering autonomous systems
\[
\begin{align*}
\mathbf{x}' &= f(x, y) \\
y' &= g(x, y)
\end{align*}
\]
where \( f \) and \( g \) are smooth everywhere. Denote a solution by \( x(t) \). We know that if \( x \) is a periodic function, meaning that for some \( T > 0 \), \( x(t + T) = x(t) \) for all \( t \), then the trajectory (or “orbit”) of \( x \) is a closed curve. Our familiar system \( x' = y, y' = -x \) has only periodic solutions, and the same is true for

\[
\begin{align*}
x' &= y \\
y' &= -x - x^3.
\end{align*}
\]

(To see this, look for an energy function.)

But the example at the beginning of section 2.2 in the previous notes, which we introduced to illustrate Hopf bifurcation, is different. Suppose that \( \mu = 1 \). Then system can be given in polar coordinates as

\[
\begin{align*}
r' &= r (1 - r^2) \\
\theta' &= 1.
\end{align*}
\]

From this we see that there is only one closed nonconstant trajectory, the curve \( x^2 + y^2 = 1 \). Further, we see that if a nonzero solution starts inside or outside this circle, then it winds in a counterclockwise direction towards the circle, from inside or outside as the case may be. This is all seen by looking at the “phase line” for \( r \). For instance, if \( 0 < r(0) < 1 \) then \( r'(0) > 0 \), and \( r(t) \) increases, tending to 1 as \( t \to \infty \).

A closed trajectory which has this property – that solutions wind onto it from either side as \( t \to \infty \) – is called a “limit cycle”. The existence of limit cycles is an important problem in ode’s.

A system can have more than one limit cycle. In class we will illustrate this with the equation whose polar form is

\[
r' = r (1 - r) (2 - r) (3 - r), \quad \theta' = 1.
\]

Again, look at the phase line. Solutions approach either \( r = 1 \) or \( r = 3 \), depending on where they start, except for the solutions with \( r = 0 \) or \( r = 2 \).

5 Van der Pol’s equation.

The van der Pol equation was introduced in notes 10, and is described in the text, starting on page 552 (example 2 in section 9.7, at least in the 8th edition) The equation is:

\[
u'' + \mu (u^2 - 1) u' + u = 0.
\]
Following our usual procedure, we set \( v = u' \) and obtain the system
\[
\begin{align*}
u' &= v \\
v' &= -u + \mu (1 - u^2) v
\end{align*}
\] (22)
This is discussed in the text, but mostly from a numerical point of view.

As always, the first step is to show that the only equilibrium point is \((0, 0)\) – this is easy – and to linearize around this point, also easy. We get the linearized matrix
\[
\begin{pmatrix}
0 & 1 \\
-1 & \mu
\end{pmatrix},
\]
with eigenvalues \( \frac{1}{2} \mu \pm \frac{1}{2} \sqrt{\mu^2 - 4} \). Thus, the origin is stable for \( \mu < 0 \) and unstable for \( \mu > 0 \). Since \( \lambda = \pm i \) when \( \mu = 0 \) it is tempting to assume that there is a Hopf bifurcation at \( \mu = 0 \), but it was seen in notes 10 that this is not the case.

If we set \( E = \frac{1}{2} (u^2 + v^2) \) then we find that
\[
\dot{E} = \mu (1 - u^2) v^2.
\]
This shows that analyzing the system may not be so easy, because \( \dot{E} < 0 \) if \( |u| > 1 \) and \( \dot{E} > 0 \) if \( |u| < 1 \). It appears possible (and indeed is the case) that solutions can cross from \( |u| < 1 \) into \( |u| > 1 \) and back again, so we do not expect the energy to be monotone.

Nullclines turn out not to be very helpful in the analysis of (22). Instead, we consider the new variables
\[
x = u \\
y = u' + \mu \left( \frac{u^3}{3} - u \right).
\]
Then we get
\[
\begin{align*}
x' &= y - \mu \left( \frac{x^3}{3} - x \right) \\
y' &= -x.
\end{align*}
\] (23)
Now the nullclines are: \( x' = 0 \) on \( y = \mu \left( \frac{x^3}{3} - x \right) \), \( y' = 0 \) on the \( y \) axis. Still, however, there is not an obvious monotone energy function.

Nevertheless, it is possible to analyze this system and show that there is a unique periodic solution for any \( \mu > 0 \). Indeed energy is used, but several energy functions are needed, with each used in only part of the phase space. The text shows some numerical solutions.

We do not have time for a thorough analysis of this system. However I have put some material in the appendix of this “chapter”. 
5.1 Second order systems are relatively “nice”.

There is an important theorem which implies that for a second order autonomous system, (sufficiently smooth), only one of the following things can happen for any particular solution $x$:

1. $x(t)$ is unbounded.
2. Along some sequence $\{t_j\}$, $x(t_j)$ tends to an equilibrium point.
3. $x(t)$ is either periodic, or its orbit tends to the orbit of a periodic solution.

This is Theorem 9.7.3 in the text (at least the 8th edition), called the “Poincaré-Bendixson theorem”. We do not have time to discuss it here. But as an example, suppose that $\mu > 0$ in (22) or (23), so that $(0, 0)$ is an unstable spiral or node. Then no solution tends to $(0, 0)$, which is the only equilibrium point. If one can show that there is a solution which is bounded, then it must tend to a periodic solution. This is an important method for showing that there is a periodic solution. It does not work for systems of three or more equations.

Another important implication of the Poincaré-Bendixson theorem is that the interesting behavior seen earlier in a dynamical system defined by a map does not happen for (22). Instead one must add a forcing term, to get a non-autonomous 2nd order equation:

$$u'' + \mu (u^2 - 1) u' + u = A \sin \omega t.$$ 

We will not go into detail about this, however. We will see such behavior also in an autonomous third order system, discussed in section 9.8 and in class.

6 Homework, due April 3

1. (10 pts.) Prove Gronwall’s lemma as stated above. Hint: Let $R(t) = \int_0^t w(s) \, ds$ and prove that $R' \leq a + bR$. It may also be helpful to observe that if we have equality in the hypotheses, then we get an ode for $R$ which can be solved exactly, giving inequality in the desired conclusion.

2. Consider the system

$$x'_1 = -x_1 - x_2^2$$
$$x'_2 = x_2 + x_1^2.$$ 

(a) (10) Use equations (13) and (21) to find $x^0$, $x^1$ and $x^2$.

(b) (10) Use the components of $x^2$ to get an approximate formula for the stable manifold at $(0, 0)$. This will be of the form $d = \alpha c^2$. It can be shown that if the
exact formula is \( d = h(c) \), then \( h(c) - \alpha c^2 = O(c^5) \) as \( c \to 0 \).

State what this means in terms of the Taylor series for \( h \).

(c)(5) Get a similar approximate formula for the unstable manifold at \((0,0)\).

(d)(10) Use PPLANE to plot the phase plane of the system above, being sure to include all equilibrium points and any bounded trajectories you can find. Include the nullclines (which can be plotted with one of the options of PPLANE). Describe any interesting features you notice, such as the “global” behavior of the different branches of the stable and unstable manifolds at \((0,0)\). Are these manifolds the graphs of functions, as they are near to \((0,0)\)?

3. (a) Use PPlane to determine the limit as \( \mu \to 0 \) of the amplitude of the limit cycle of van der Pol’s equation. Include a plot of the limit cycle when \( \mu = .1 \).

(b) Use PPLANE to give a plot of the limit cycle of van der Pol’s equation when \( \mu = 50 \). Include the nullclines in your plot. (These can also be found using the "solution" tab.)

When you click on a point in the window PPLANE graphs the trajectories both forward and backward, and for part (b) usually gives extraneous sections of the solution you clicked on, which are obviously not part of the limit cycle. In addition to any printouts you may submit, include a hand drawing showing just the limit cycle and the nullclines.

(c) Describe what “qualitative” differences you see between the limit cycles in (a) and (b). Sometimes the case \( \mu = 50 \) is called a “fast-slow” system. Explain what is meant by this.

6.1 Appendix A

We now consider van der Pol’s equation in the case where \( \mu \) is positive but small. We would expect that the solutions are somehow close to solutions when \( \mu = 0 \). But then we have circles in the phase plane, of any radius. In other words, there are infinitely many periodic solutions, all of period \( 2\pi \). For \( \mu > 0 \) we stated that there

---

\( ^1 \) This notation was discussed in class.

Definition: If \( f \) and \( g \) are defined in some neighborhood of \( 0 \) in \( \mathbb{R} \), then we say \( f(x) = O(g(x)) \) as \( x \to 0 \) if there is a \( \delta > 0 \) such that \( \frac{f(x)}{g(x)} \) is bounded in the set \( 0 < |x| < \delta \). As examples, \( \sin x = O(x) \) as \( x \to 0 \), but \( \sin x \) is not \( O(x^2) \) as \( x \to 0 \). Also, (extending the notation a bit), \( x^2 + 1 = O(x^2) \) as \( x \to \infty \).
was a unique periodic solution. We can ask what happens to this solution as \( \mu \to 0 \). We suppose that it must approach some circle, and we can ask which circle – with which radius.

In the text graphs are shown which suggest that, somewhat surprisingly, the limiting periodic solution as \( \mu \to 0^+ \) is the one whose graph is a circle of radius 2.

The first comment to make is that this is obviously not a Hopf bifurcation, because that requires that for small \( \mu \) the amplitude of the periodic solution is proportional to \( \sqrt{\mu} \), and in particular tends to zero as \( \mu \to 0 \). But the really interesting question is: Why is the limiting radius equal to two?

In fact, many books omit the proof of this fact, or don’t even mention it. Here I will give proof taken from the book Nonlinear Ordinary Differential Equations, by R. Grimshaw.

In this proof I will assume that use of several energy functions, as described above, has allowed us to prove the existence of a unique periodic solution \((x(t, \mu), y(t, \mu))\) to (23) and moreover, to show there is an \( M > 0 \) such that for all \( \mu \), \( x^2 + y^2 \leq M \) on this periodic solution. In other words, the solutions are uniformly bounded.

We can also assume, from a phase plane argument, that \( x(0, \mu) = 0 \) for each \( \mu \).

Consider any sequence \( \{ \mu_j \} \) of positive numbers which converges to zero. Since \( \{y(0, \mu_j)\} \) is bounded, there is a subsequence of \( \{\mu_j\} \) such that \( y(0, \mu_j) \) converges, say to some \( r_0 > 0 \). Then the general theory of ode’s which we gave earlier in the course can show that along this sequence, the solution \((x(t, \mu_j), y(t, \mu_j))\) tends to \((r_0 \sin t, r_0 \cos t)\) uniformly on the interval \([0, 4\pi]\). Further, if \( T_j \) is the period of \((x(t, \mu_j), y(t, \mu_j))\), then \( T_j \to 2\pi \) as \( j \to \infty \) (along the subsequence).

Let \( E(x, y) = \frac{1}{2}(x^2 + y^2) \). Then from (23) it follows that if \((x(t), y(t)) = (x(t, \mu_j), y(t, \mu_j))\), then

\[
\frac{d}{dt} \left( \frac{1}{2} (x(t) + y(t)^2) \right) = x(t)^2 \left( 1 - \frac{1}{3} x(t)^2 \right).
\]

Since the solution is periodic, \( E \) must come back to where it started at \( t = T \), and so we get

\[
\int_0^T x(t)^2 \left( 1 - \frac{1}{3} x(t)^2 \right) dt = 0.
\]

But we have seen that \((x(t), y(t)) \to (r_0 \sin t, r_0 \cos t)\) as \( j \to \infty \), uniformly on \([0, 4\pi]\), and hence,

\[
\int_0^{2\pi} r_0^2 (\sin^2 t) \left( 1 - \frac{1}{3} r_0^2 \sin^2 t \right) dt = 0.
\]
But this integral can be evaluated explicitly, and from this we get $\pi r_0^2 - \frac{1}{2}\pi r_0^4 = 0$. We therefore see that $r_0 = 2$. This would be true for any subsequence along which $x(t, \mu_j)$ converges, so in fact the original sequence must converge, proving the result.

The theory of van der Pol’s equation is also interesting when $\mu$ is a large positive number. Here I will simply show plots of $u(t)$ for $\mu = .1$ and $\mu = 10$, and also phase plots for these values. It is interesting to consider why these graphs look as they do.

![Graphs of van der Pol's equation](image)

### 6.2 Appendix B

This starts from equation (21). Suppose that $x^j$ has been defined, and

$$|x^j(t)| \leq \rho_0 e^{-(1-\epsilon)t}$$

for $t \geq 0$. Then the steps leading to (20) can be repeated with $x^j$ substituted for $x^0$ to show that $x^{j+1}$ is defined and also satisfies (24).

In particular, each $x^j_i(t)$ is bounded by $\rho_0$ on $[0, \infty)$ and tends to zero at an exponential rate as $t \rightarrow \infty$.

Also, the Lipschitz condition continues to apply to the difference between the successive approximations. We therefore have

$$|x_1^{j+1}(t) - x_1^j(t)| \leq e^{-t} \int_0^t e^s L(\rho_0) |x^j(s) - x^{j-1}(s)| \, ds$$

and

$$|x_2^{j+1}(t) - x_2^j(t)| \leq e^{t} \int_0^\infty e^{-s} L(\rho_0) |x^j(s) - x^{j-1}(s)| \, ds.$$

The inductive steps are then almost the same as those defining the $x^j$. We find that if

$$|x^j(t) - x^{j-1}(t)| \leq \left(\frac{1}{2}\right)^{j-1} \rho_0 e^{-(1-\epsilon)t}$$
for \( t \geq 0 \), then
\[
|x^{j+1}_1(t) - x^j_1(t)| \leq e^{-t} \int_0^t e^{s} L(\rho_0) \left( \frac{1}{2} \right)^{j-1} \rho_0 e^{-(1-\varepsilon)s} ds \
\leq e^{-(1-\varepsilon)t} \left( \frac{1}{2} \right)^{j-1} \rho_0 \frac{L(\rho_0)}{\varepsilon} \leq \left( \frac{1}{2} \right)^{j} \rho_0 e^{-(1-\varepsilon)t}
\]
because of (19). Also,
\[
|x^{j+1}_2(t) - x^j_2(t)| \leq e^{t} \int_t^\infty e^{-s} L(\rho_0) \left( \frac{1}{2} \right)^{j-1} \rho_0 e^{-(1-\varepsilon)s} ds \
\leq \left( \frac{1}{2} \right)^{j-1} \rho_0 \frac{L(\rho_0)}{2 - \varepsilon} e^{-(1-\varepsilon)t} \leq \left( \frac{1}{2} \right)^{j} \rho_0 e^{-(1-\varepsilon)t}.
\]
Because \( \sum_{j=1}^\infty (1/2)^j \) converges to 1, it then follows that \( \sum_{j=1}^\infty (x^j - x^{j-1}) \) converges uniformly in \([0, \infty)\), and \( x^0 + \sum_{j=1}^\infty (x^j - x^{j-1}) \) converges to a solution of the equations (11) and (12) as desired. Further, this solution satisfies
\[
|x(t)| \leq \rho_0 e^{-(1-\varepsilon)t}
\]
and so tends to \((0,0)\) as \( t \to \infty \). This completes the proof of Theorem 1.