Math 1180, Hastings

2009 notes, part 8,

Determinants

We will define determinant differently from the book, though it follows from material in the book that the two definitions are entirely consistent. We define the determinant of a square matrix A to be a real valued function of A which satisfies certain properties, which, as you will see, are related to Gaussian elimination.

Definition 1 *The determinant of a square matrix is the scalar quantity which is defined by the following four properties:*
(i) Adding a multiple of one row to another does not change the determinant
(ii) Interchanging two rows changes the sign of the determinant
(iii) Multiplying ***one*** *row of a matrix by a scalar c multiplies the determinant by c*
(iv) The determinant of the identity is 1.

Please note that in (i), we add a multiple of row (i) to row (j). This changes row (j). It does not change row (i).

This "definition" has a problem. It can only be considered a definition if there is only one function with these properties. Unfortunately we do not have time to prove this theorem, although I believe that arguments in the text can be put together to prove it.

From this definition we see that we can evaluate determinants using Gaussian elimination. This is the best way for most determinants that are more than 3 × 3, though we will see some exceptions below.

We start with the 2 × 2 matrix, where you probably know the formula:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Let's check the properties:

$$\det\begin{pmatrix} a & b \\ c+ka & d+kb \end{pmatrix} = a\,(d+kb) - b\,(c+ka) = ad + akb - bc - bka = ad - bc = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$$\det \begin{pmatrix} c & d \\ a & b \end{pmatrix} = cb - ad = -(ad - bc)$$

$$\det \begin{pmatrix} ka & kb \\ c & d \end{pmatrix} = kad - kbc = k(ad - bc).$$

$$\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 - 0.$$

Thus, all the properties are satisfied. From now on we will use the rules to evaluate the determinant, if the matrix is bigger than 2 × 2.

Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ 3 & 8 & 10 \end{pmatrix}.$$

We use Gaussian elimination to reduce this to the identity matrix. This is the same as what we did when finding the inverse, except that we don't have to keep track of what happens to I under these operations. Some of the steps change the determinant, some do not. Let us suppose that the determinant is D. We do the Gaussian elimination, keeping track of what happens to D. We are helped in this by the fact that the most complicated of the elementary operations, adding a multiple of one row to another, does not change the determinant at all, and the other two change it in simple ways.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ 3 & 8 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$D \to D \to -D$$

$$\to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\to -\frac{1}{2}D \to -\frac{1}{4}D \to -\frac{1}{4}D.$$

I moved rather fast in the last step, combining three operations into one step. But it is easy to see that this is correct because of the $1's$ down the diagonal. Using our axiom (iv) we see that

$$-\frac{1}{4}D = 1,$$

and so $D = -4$.

But we don't have to do quite so many steps. If we stopped at the end of the first line above, when we had an upper triangular matrix, then we see that $-D$ is the product of the diagonal elements

Here are some important properties that follow from the definition;

Proposition 2 *(i) The determinant of an upper triangular matrix (including diagonal matrices) is the product of the diagonal elements.*

(ii) If some pivot in the echelon form is zero, the determinant is zero

(iii) If two rows are equal, or if one row is a scalar multiple of another row, then the determinant is zero.

See if you can prove these.

Here is a relatively simple larger example:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \\ 0 & -12 & -24 & -36 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so the determinant is zero. Here we applied the rule that if two rows are equal, then the determinant is zero. But also, this rule implies that if just one row is all zeros, then the determinant is zero, because we could add some nonzero row to the zero row and get two rows equal.

Now we prove one of the important properties that can be derived from the definition. This is the following:

Theorem 3 If A *and* B *are* $n \times n$ *matrices, then*

$$\det AB = \det A \det B \tag{1}$$

.

Proof. We have two cases to consider. Either (i) $\det B = 0$ or (ii) $\det B \neq 0$.In case (i) we recall that $\det B = 0$ implies that the echelon form of B has a zero along the diagonal, or in other words, a zero pivot. This implies that B is singular. We saw earlier that in this case, the equation

$$B\mathbf{x} = \mathbf{0}$$

has a nontrivial solution. If $\mathbf{v}$ is that solution, then

$$AB\mathbf{v} = A(B\mathbf{v}) = A\mathbf{0} = \mathbf{0},$$

which implies that the matrix AB is singular. This in turn implies that AB has a zero pivot and so, as we just saw, $\det AB = 0$. This proves (1) in the case where $\det B = 0$.

Now consider case (ii), $\det B \neq 0$. In this case, we define a function of matrices as follows:

$$f(A) = \frac{\det AB}{\det B}. \tag{2}$$

Our goal is to show that $f(A)$ has the four properties listed above for the determinant function, and therefore that $f(A)$ **is** the determinant function. This will verify formula (1). One of the properties is easy to verify: $f(I) = \frac{\det IB}{\det B} = \frac{\det B}{\det B} = 1$.For the other properties we observe that if E is an elementary matrix, then $(EA)B = E(AB)$. Hence,

$$f(EA) = \frac{\det((EA)B)}{\det B} = \frac{\det(E(AB))}{\det B}.$$

If E is the matrix for adding a multiple of one row of a matrix to another, then $\det(E(AB)) = \det AB$ and so $f(EA) = f(A)$. If E is the matrix for multiplying a row of a matrix by a constant c, then it multiplies the determinant of AB times c, and so $f(EA) = cf(A)$. If it is an elementary permutation matrix then $\det(E(AB)) = -\det(AB)$ and so $f(EA) = -f(A)$.

This verifies all the properties of determinant, so $f(A)$ must be the determinant of A, verifying formula (2). ■

Proposition 4 *(iv)* $\det A = \det A^T$

(v) The determinant can also be evaluated by "elementary column operations"

(vi) If two columns are equal, or one is a scalar multiple of another, the determinant is zero

(vii) $\det A = 0$ *if and only if* A *is singular.*

3×3 **determinants** There is a method for 3×3 determinants which is sometimes helpful. The problem with this method is that students sometimes try to use it with 4×4 's, where it gives the wrong answer

Use this method only for three by three determinants.

Method: Copy the determinant to the right of itself:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Now imagine a diagonal line down the diagonal of the left determinant, through the a, e, i. Move this line to the right, and using the second one, the line goes through b, f, g. Do this again and it goes through c, d, h.

Now take the other diagonal of the first copy, through c, e, g, move this to the right, through a, f, h, and again, through c, e, g. Then the determinant is found by taking each of the products of the three terms, but with a sign change as follows:

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - ceg - afh - bdi..$$

If you find this confusing, look at the diagrams of the method on pg. 174 of the text.

Example: Find $\det \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ 3 & 8 & 10 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ 3 & 8 & 10 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ 3 & 8 & 10 \end{pmatrix}.$$

$$40 + 48 + 48 - 36 - 64 - 40 = -4$$

:Check: This was the same 3×3 as we did earlier, where we got the same answer.

If there are some negative entries, keeping the signs straight gets harder.

Cofactors and minors.

There is another way to compute a determinant which is sometimes helpful. This method is very inefficient for a large matrix, and is never used except in smaller matrices where we want to do the computation “by hand”. But it is sometimes useful theoretically.

Definition: Let A be an $n \times n$ matrix with entries a_{ij}. Let A_{ij} denote the $(n-1) \times (n-1)$ matrix obtained by crossing out the i^{th} row and j^{th} column of A. Then the “ij^{th} cofactor” of A is

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

We then have

Proposition 3.1 (Same as Definition 4, page 176 of the text, but the notation is different there. I used “A_{ij}” differently.) For any i :

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} \tag{3}$$

This is called ”expansion of $\det A$ by the i^{th} row. By taking the transpose of A, and recalling that $\det A = \det A^T$, we see that we can also expand by the i^{th} column:

$$\det A = \sum_{j=1}^{n} a_{ji} C_{ji}$$

We can use this to choose the best row or column for the calculation, often a row or column with as many zeros as possible.

The signs of the terms is a source of confusion. The C_{ij} are $(-1)^{i+j}$ times the determinant of a smaller matrix. So C_{11} is the smaller determinant, while C_{12} is minus the smaller determinant. Look at the examples below.

The first example is the same three by three as before. It doesn't make much difference which method you use here, but I want to illustrate expansion by the first row, and then expansion by the second column. Be sure you understand each step.

$$\begin{aligned} A &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ 3 & 8 & 10 \end{pmatrix} \\ \det A &= 1 \det \begin{pmatrix} 4 & 8 \\ 8 & 10 \end{pmatrix} - 2 \det \begin{pmatrix} 2 & 8 \\ 3 & 10 \end{pmatrix} + 3 \det \begin{pmatrix} 2 & 4 \\ 3 & 8 \end{pmatrix} \\ &= -24 - 2(-4) + 3(4) = -4 \end{aligned}$$

$$\begin{aligned}\det A &= (-2)\det\begin{pmatrix} 2 & 8 \\ 3 & 10 \end{pmatrix} + 4\det\begin{pmatrix} 1 & 3 \\ 3 & 10 \end{pmatrix} - 8\det\begin{pmatrix} 1 & 3 \\ 2 & 8 \end{pmatrix} \qquad (4)\\ &= -2\,(-4) + 4\,(1) - 8\,(2) = -4. \qquad (5)\end{aligned}$$

Here's one where the correct method is obvious.

$$A = \begin{pmatrix} 1 & 3 & 2 & -5 \\ 0 & 0 & 6 & 0 \\ 2 & 1 & 2 & 5 \\ -3 & 0 & 1 & 1 \end{pmatrix}$$

Solution: Expand by the second row. (Be sure you see where the minus sign in front of the 6 comes from.) Then expand the 3 × 3 by the second column. Again, watch the signs.

$$\begin{aligned}\det A &= -6\det\begin{pmatrix} 1 & 3 & -5 \\ 2 & 1 & 5 \\ -3 & 0 & 1 \end{pmatrix}\\ &= -6\left(-3\det\begin{pmatrix} 2 & 5 \\ -3 & 1 \end{pmatrix} + 1\det\begin{pmatrix} 1 & -5 \\ -3 & 1 \end{pmatrix}\right) \qquad (6)\\ &= -6\,(-3\,(17) + (-14)) = 390 \qquad (7)\end{aligned}$$

Let's check:

$$\det\begin{pmatrix} 1 & 3 & 2 & -5 \\ 0 & 0 & 6 & 0 \\ 2 & 1 & 2 & 5 \\ -3 & 0 & 1 & 1 \end{pmatrix} : 390$$

Hmm. What kind of a check is that? Hint: the best linear algebra software package is Matlab, which is widely used in the engineering school, and also in math. But Maple and Mathematica can easily do this calculation as well. So can graphics calculators.

Efficient computation of determinants.

For **3** × **3**, use the "diagonal" method described above, unless one row or column has two zeros. Then use cofactor expansion.

For 4 × 4, if any row or column has at least two zeros, consider cofactor expansion. Or try Gaussian elimination in order to get a row or column with two or three zeros and then consider cofactor expansion. Often using Gaussian elimination to get to upper triangular form is best. (Then take the product of diagonals.) (You need to keep track of changes caused by the Gaussian elimination steps.

For larger matrices, use a computer unless you see a quick way via cofactors. There need to be a number of zeros strategically placed to get anywhere by hand.

Computer programs used methods based on Gaussian elimination.

Here is some data I found: If a matrix is $n \times n$, Gaussian elimination takes around $\frac{2}{3}n^3$ arithmetic steps (addition and multiplication), while using cofactors takes over $n!$ steps. So let's suppose the matrix is 100×100. Gaussian elimination might take fewer than one million steps. Using cofactors takes at least 10^{158} steps. Let's see. I read that the fastest current computer does "20 quadrillion calculations per second". I think a quadrillion is 1,000,000,000,000,000 or $10^{15.}$ Allowing for the 20, and being generous, let's say 10^{17}. So, we need 10^{141} seconds. This will take awhile. The universe is said to be less than 10^{19} seconds old.

How long did Gaussian elimination take? 10^6 steps at more than 10^{16} steps per second, or 10^{-10} seconds. So the ratio between the two methods is about 10^{151}. Any guesses on my recommendation?

Further properties of determinants

Proposition 5 *(iv)* $\det A = \det A^T$

(v) The determinant can also be evaluated by "elementary column operations"

(vi) If two columns are equal, or one is a scalar multiple of another, the determinant is zero

(vii) $\det A = 0$ *if and only if* A *is singular.*

Proof. (iv) can be proved by showing that the function $g(A) = \det\left(A^T\right)$ has the properties of the determinant function, and using the fact (not proved yet!) that there is only one function with these properties.

(iv) implies (v) and (vi).

For (vii) we observe that we evaluate the determinant by doing row operations to get A into row echelon form. If there are n pivots, then they must all be on the diagonal, giving a nonzero determinant. If there are fewer than n pivots, then one of the diagonal elements of the row echelon form must be zero, giving a zero determinant. We already saw that fewer than n pivots implies that A is singular, and has no inverse. ∎

Homework: Due Wednesday, Oct. 9.

1. Find

$$\det\begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 2 & 4 & 1 & 2 & 3 \\ 1 & 3 & 2 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}$$

Hint: A combination of the methods may be the best approach in a hand calculation.

2. Each of the integers 1898, 3471, 7215, and 8164 is divisible by 13. Without expanding the determinant, prove that

$$\det\begin{pmatrix} 1 & 8 & 9 & 8 \\ 3 & 4 & 7 & 1 \\ 7 & 2 & 1 & 5 \\ 8 & 1 & 6 & 4 \end{pmatrix}$$

is divisible by 13. Hint: Work with the transpose.