Math 1180, fall, 2009

notes #5

1 Vectors and Matrices

Read in text: pgs 45-53, 61-66, 69-72.

We have already defined a matrix, as a rectangular array of numbers

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1\ (n-1)} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2(n-1)} & a_{2n} \\ \cdots & \cdot & \cdots & \cdot & \cdots \\ a_{(m-1)1} & \cdot & \cdots & \cdot & a_{(m-1)n} \\ a_{m1} & \cdot & \cdots & \cdot & a_{mn} \end{pmatrix}.$$

Sometimes we will refer to the matrix A as

$$(a_{ij}).$$

We define two kinds of vectors. A **row vector** is a matrix with only one row. ($m = 1$.) A column vector is a matrix with only one column. ($n = 1$.) We will deal most often with column vectors. Usually we will say just "vector", and this will mean a column vector. If we need to discuss row vectors, we will refer to them as "row vectors". A column vector will be denoted by an bold face lower case letter, such as

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

(In class, vectors will be underlined.)

We now discuss the "arithmetic" of matrices (and hence also of vectors, since vectors are special kinds of matrices. This means, we define addition, subtraction, multiplication and division of matrices. Actually, division is not defined. The text defines what it means to add and multiply matrices, with subtraction implicit under addition. It also defines "scalar multiplication".

See page 51 and 52 for addition and scalar multiplication, with several examples. Here I will simply give a few examples.

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} + \begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix} = \begin{pmatrix} a+u & b+v & c+w \\ d+x & e+y & f+z \end{pmatrix}. \tag{1}$$

In other words, addition is done "component wise".

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & -5 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 9 \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} 2 & 3 \\ 4 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \text{ is not defined, because they are different sizes.} \tag{3}$$

$$3\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 2\begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 12 \\ 13 \end{pmatrix} \quad \text{(check)}$$

This last example leads to an important definition. In this definition, R^n denotes the set of all vectors $\begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}$ with n components.

Definition 1 *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are vectors in* R^n *and* a *and* b *are real numbers, then the vector*

$$a\mathbf{x}+b\mathbf{y}$$

is called a "linear combination" of $\mathbf{x}$ *and* $\mathbf{y}$*. More generally, if* $\mathbf{x}^1, ..., \mathbf{x}^k$ *are all vectors in* R^n*, and* $c_1, ...c_k$ *are scalars (real numbers), then*

$$c_1\mathbf{x}^1 + c_2\mathbf{x}^2 + \cdots + c_k\mathbf{x}^k$$

is called a "linear combination of $\mathbf{x}^1, ..., \mathbf{x}^k$*.*

We will frequently refer to linear combinations of vectors.

Multiplication of matrices is more complicated. The text starts by defining a product of vectors. As you have probably seen in calc III, there are two different products of vectors, the "cross product" and the "dot product". We will not use the cross product in this course. We will often use the dot product, which is also called the "scalar product".

Definition 2 *The dot product, also called the "inner" product, of two vectors* x *and* y *in* R^n *is*

$$\mathbf{x}\cdot\mathbf{y}=\begin{pmatrix} x_1\\ x_2\\ \cdots\\ x_n\end{pmatrix}\cdot\begin{pmatrix} y_1\\ y_2\\ \cdots\\ y_n\end{pmatrix}=x_1y_1+x_2y_2+\cdots+x_ny_n. \tag{4}$$

Example:

$$\begin{pmatrix} 3\\ 2\\ -1\\ 5\end{pmatrix}\cdot\begin{pmatrix} 1\\ 3\\ -4\\ 2\end{pmatrix}=(3)\,(1)+(2)\,(3)+(-1)\,(-4)+(5)\,(2)=3+6+4+10=23. \tag{5}$$

Note: The dot product of two vectors is a scalar, not a vector.

This is frequently forgotten by students. If I see something like

$$\begin{pmatrix} 1\\ 3\end{pmatrix}\cdot\begin{pmatrix} 2\\ 5\end{pmatrix}=(2,15)\,, \quad \text{or} \quad \begin{pmatrix} 2\\ 15\end{pmatrix},$$

in an exam, no credit will be given for that problem. The correct answer is 17.

Several properties of dot product are proved in the book. For example,

$$\mathbf{x}\cdot(\mathbf{y}+\mathbf{z})=\mathbf{x}\cdot\mathbf{y}+\mathbf{x}\cdot\mathbf{z},$$

which is the distributive property. Also,

$$\mathbf{x}\cdot\mathbf{y}=\mathbf{y}\cdot\mathbf{x},$$

which is the commutative property. However, $\mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{c})\neq(\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c}$. Why not?

On page 64 of the text, the product of two matrices is defined in terms of dot product. Here I will give an equivalent formula.

Definition 3 *If A is an $m\times n$ matrix and B is a $p\times q$ matrix, then AB is an $m\times q$ matrix, and if $C=AB$ has components c_{ij} , then*

$$c_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj}.$$

I will give some examples in class. The matrix product satisfies most of the usual properties of arithmetic, such as

$$A(BC) = (AB)C \quad \text{– the associative property}$$
$$A(B+C) = AB + AC \text{– the distributive property}$$

However, the commutative property does not apply to matrices. Recall from class that if A is $m \times n$ and B is $p \times q$, then AB is defined only if $n = p$. But even if AB is defined, BA might not be. That would require that $m = q$. Note also that if A is $m \times n$ and B is $n \times q$, then AB is $m \times q$. Symbolically, I will often write in class:

$$(m \times n)(n \times q) = m \times q.$$

Examples will be given in class.

But even if A and B are both $n \times n$, so that AB andBA are both defined and of the same size, most of the time, $AB \neq BA$, as in this example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 7 & 7 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 4 & 6 \end{pmatrix}.$$

Hence, the algebra of matrices is quite different from the algebra of scalars. This has important consequences.

Next, suppose A is $m \times n$ and $\mathbf{x} \in \mathbf{R}^n$. This means that $\mathbf{x}$ is an $n \times 1$ matrix, and so $A\mathbf{x}$ exists, and is an $m \times 1$ matrix.

See how this fits in with our notation

$$A\mathbf{x} = \mathbf{b}$$

for a system of m equations and n unknows.

Theorem 4 *If A is an $m \times n$ matrix, $\mathbf{b}$ is a vector in R^m, and $A\mathbf{x} = \mathbf{b}$ has a solution, then $\mathbf{b}$ is a linear combination of the columns of A.*

Proof. We consider the product $A\mathbf{x}$, where $A = (a_{ij})$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix}$:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1\ (n-1)} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2(n-1)} & a_{2n} \\ \cdots & \cdot & \cdots & \cdot & \cdots \\ a_{(m-1)1} & \cdot & \cdots & \cdot & a_{(m-1)n} \\ a_{m1} & \cdot & \cdots & \cdot & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \cdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \cdots \\ a_{mn} \end{pmatrix}.$$

This is a linear combinations of the columns of A . ■

We can extend this idea to a matrix product AB, where A is $m \times n$ and B is $n \times q$. By looking at the formula for matrix multiplication, we can see that if $C = AB$, then

1. The first column of C is the product Ab^1, where $\mathbf{b}^1$ is the first column of B. Hence the first column of C is a linear combination of the columns of A.

2. **Each** column of C is a linear of the columns of A. The j^{th} column of C is the product $A\mathbf{b}^j$, where$\mathbf{b}^j$ is the j^{th} column of $B/$

As an example, calculate that

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

Check the first column of C:

$$5 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 7 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 19 \\ 43 \end{pmatrix}, \qquad (6)$$

while the second is

$$6 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 8 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 22 \\ 50 \end{pmatrix}.$$

Homework:

1.1 Inverse of a matrix.

There is no definition of matrix division. But there is a definition of inverse, for some **square** matrices. For this definition, we let I denote the $n \times n$ matrix with 1 on the diagonal, 0 elsewhere. Note the following property: If A is any matrix with n rows, then

$$IA = A.$$

If A has n columns, then

$$AI = A.$$

Definition 5 ***If*** *A and B are $n \times n$ matrices, and $AB = BA = I$, then B is called the inverse of A and A is called the inverse of B.*

We write

$$\begin{aligned} A &= B^{-1} \\ B &= A^{-1}. \end{aligned}$$

Example:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In fact, we can show that if A and B are square and the same size, and $AB = I$, then automatically, $BA = I$.

But there are many matrices which have no inverse. For example, consider

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Suppose that $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $AB = I$. Then,

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

or

$$\begin{aligned} a + c &= 1 \\ b + d &= 0 \\ a + c &= 0 \\ b + d &= 1 \end{aligned}$$

There are two contradictions, so no inverse exists. We will learn several ways to determine if an inverse exists for a square matrix.

Definition 6 *If A is an $n \times n$ matrix and A^{-1} exists, then A is called "non-singular". If A^{-1} doesn't exist, then A is called "singular".*

Here is a sample result one can prove using the inverse.

Proposition 7 *If A is an $n \times n$ nonsingular matrix, and B and C are $n \times q$ matrices, and if*

$$AB = AC, \tag{7}$$

then $B = C$.

Proof. Multiply each side of (7) by A^{-1}. We get

$$\begin{aligned} A^{-1}(AB) &= A^{-1}\left(AC \right) \\ \left(A^{-1}A\right)B &= \left(A^{-1}A\right)C \\ IB &= IC \\ B &= C \end{aligned}$$

■

Homework: Due Wednesday, September 23 at the beginning of class.

1. Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 3 \end{pmatrix}.$$

Calculate each of the following, or explain why it is not defined.

$$\begin{aligned} &2A - B \\ &C + D \\ &CD \\ &DC \end{aligned}$$

2. (a) Suppose that $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$. The "transpose" of $\mathbf{x}$ is defined to be the row vector $\mathbf{x}^T = (x_1, x_2, x_3)$. Prove that

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y},$$

where the left side is the usual dot product, and the right side is the usual matrix product. Also compute the matrix product $\mathbf{y}\mathbf{x}^T$, if this product is defined.

(b) pg. 80, # 40 (See definition just before problem 35).

3. Pg. 82, #52. This is your first proof. See if you can do it efficiently.

4. With the same A, B as in problem 1, determine if

$$(A+B)(A-B) = A^2 - B^2.$$

Explain the answer you get, in light of the standard equation from elementary algebra

$$(a+b)(a-b) = a^2 - b^2.$$

5. Find 2×2 matrices A, B, C such that $AB = CB$ and $B \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, but $A \neq C$. Hint: Look for the simplest example you can, where the matrices only have 0's and 1's as entries.