Math 1180, fall, 2009

notes #4

In the last notes we defined a "pivot" for a matrix A as the first nonzero element in a row of a row echelon form of A. Here is an important fact, which we state as our first theorem.

Theorem 1 *If A is in row echelon form, and B is obtained from A by a Gaussian elimination step, with $B \neq A$, then either B is not in row echelon form, or B has pivots in exactly the same locations as A.*

Proof. *Multiplying a row by a nonzero number does not change the status of a pivot. Interchanging two rows must destroy the property of being in row echelon form, unless both rows are zero, in which case $B = A$, so the interchange operation is not allowed in going from A to B. Adding a multiple of a higher row to a lower row also changes the echelon status, since the first nonzero element of the higher row now has a nonzero element below it. So that step is not allowed. Adding a multiple of a lower row to a higher row does not change the location of the pivot in the higher row, since 0 is added to the pivot of the higher row. These are all the Gaussian elimination steps, so this proves the theorem.* ■

This result suggests that all row echelon forms of A have pivots in exactly the same locations. It further suggests that there is only one reduced row echelon form. However, if you think about it, we have not proved this. In fact, it is tricky to prove, and requires mathematical induction. We will not give the proof, nor is it in the text.

Definition 2 *The **rank** of A is the number of pivots in any row reduced echelon form of A.*

The rank of a matrix is not defined in the text until page 349. However it is important in deciding how many solutions a system has, so I am introducing it earlier.

We usually denote the rank of A by r. We have three important numbers to consider: m, n, r. These help determine how many solutions a system has.

Theorem 3 *For any $m \times n$ matrix A, $r \leq n$ and $r \leq m$.*

Proof. There can be no more than one pivot in any row, by definition of pivot. There also can be no more than one pivot in any column, because in reduced echelon form, the entries directly below any pivot are all zero, and all the higher pivots are to the left. ∎

Definition 4 *A pivot column of A is a column that contains a pivot in some row echelon form of A. A non-pivot column is a column that contains no pivots.*

Now for some examples. Suppose that **after doing the Gaussian elimination**, you end up with

$$(A|\mathbf{b}) = \begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

The A part is

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and this is in reduced row echelon form. We have $m = 3, n = 4, r = 3$. The augmented matrix now is equivalent to the equations

$$\begin{aligned} x_1 + 2x_3 &= 1 \\ x_2 + 2x_3 &= 2 \\ x_4 &= 1. \end{aligned}$$

The pivot columns are x_1, x_2, and x_4. The non-pivot column is x_3. We get:

$$\begin{aligned} x_4 &= 1 \\ x_2 &= 2 - 2x_3 \\ x_1 &= 1 - 2x_3. \end{aligned}$$

The general solution is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ -2 \\ 1 \\ 0 \end{pmatrix}.$$

Sometimes there is no solution. You know this if, when you reach the row echelon form, the A part has an all zero row, and the b part is not zero in that row. Here is an example. Suppose that **after Gaussian elimination** you reach the matrix

$$(A \mid \mathbf{b}) = \begin{pmatrix} 1 & 3 & 4 & 2 & 0 \\ 0 & 3 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

Since this is the augmented matrix, it represents a system of four equations in four unknowns. The last equation is

$$0 = 3.$$

Obviously there is no solution. In this example, $m = 4, n = 4, r = 3$.

If you have more equations than unknowns, $(m > n)$, then you must have some all zero rows for the A part. Why? Usually the $\mathbf{b}$ part will not be zero in those rows and there will be no solution. But sometimes the $\mathbf{b}$ part could be zero in all of those rows, and so there would be a solution. Here are two examples, showing the two cases. Each one is **after** reduction to row echelon form, and each of them is an augmented matrix, so that $m = 5, n = 4, r = 4$.

$$\begin{pmatrix} 1 & 6 & 4 & 3 & 0 \\ 0 & 5 & 8 & 1 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 6 & 4 & 3 & 1 \\ 0 & 5 & 8 & 1 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{1}$$

(How many solutions are there for each system?)

Suppose that $m < n$. In this case, since there is no more than one pivot in any row, we cannot have n pivots. But it might still be possible reach the "0 = 1" sort of situation. An example is

$$(A|\mathbf{b}) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that $m = 2, n = 3, r = 1$, and there are no solutions. The equations are inconsistent.

Theorem 5 *If $r = m = n$, then $A\mathbf{x} = \mathbf{b}$ has exactly one solution, for any $\mathbf{b}$. If $r = n < m$, then there is either no solution, or a unique solution. If $r < n$, then there is either no solution, or infinitely many solutions.*

Proof. *Suppose that the augmented matrix $(A|\mathbf{b})$ has an echelon form $(A^*, \mathbf{b}^*)$. If $r = m = n$, then there is a pivot in each row and column of A^*. Starting in the last column, this allows us to find a unique x_n, then a unique x_{n-1}, and so forth, giving a unique solution.*

If $r = n < m$, then A^* has some all zero rows. Then whether there is a solution at all depends on whether the elements of $\mathbf{b}^*$ are zero in the rows corresponding to the zero rows of A^*.

If $r < n$, then some variables cannot be solved for in terms of the later variables. (For example, we might not be able to solve for x_2 in terms of x_3 and x_4, because x_2 is a nonpivot variable. So we are free to choose x_2 arbitrarily, and then solve for x_1 in terms of x_2.) These are called "free variables. They are the parameters in the solution. If $r = m$, then there is one pivot in each row, and we can solve for each of the m pivot variables, in terms of the nonpivot variables. Since the nonpivot variables can be chosen arbitrarily, there are infinitely many solutions. However if $r < m$ then there will be some all zero rows, and again, whether there are solutions depends on whether $\mathbf{b}^*$ is zero in those rows. ■

0.1 Homogeneous systems.

An important special case which we should consider in detail is when $\mathbf{b} = \mathbf{0}$. This is called the "homogeneous" case. In this case we do not need an augmented matrix. There is no point in repeatedly adding and subtracting zeros.

In the homogeneous case, there is always a solution, whatever m and n may be and whatever the matrix is.

This solution is simply $\mathbf{x} = \mathbf{0}$. The question of interest is whether there are solutions which are not zero, and this we now discuss.

1 Homogeneous case

Assume therefore that we have a homogeneous system of equations

$$A\mathbf{x} = \mathbf{0}$$

with m equations and n unknowns. This means that the matrix A (**unaugmented**) will have m rows and n columns. We do Guassian elimination until we reach row echelon form, and we determine the number of pivots.

Hence there are the following possibilities for a homogeneous system.

1. $r = n$. Then there is a pivot in every column. This means that we can solve **uniquely** for $x_1, ..., x_{n.}$ (We start with x_n.) Since $\mathbf{x} = \mathbf{0}$ is a solution, it must be the only solution. There are no nonzero solutions.

An example already in row echelon form is

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

2. $r < n$. Then those variables whose corresponding column has no pivot can be chosen arbitrarily, since there is no equation giving them in terms of the later variables. There must be an infinite number of solutions.

An example, also in row echelon form, is

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

These are the only possibilities in the homogeneous case. Either there is one solution, $\mathbf{x} = \mathbf{0}$, or there are infinitely many solutions.

Notice that it doesn't matter what m is, except that if $m < n$, then we cannot have $r = n$. There must be an infinite number of solutions. An example:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Here x_3 is a free variable and the general solution is

$$\mathbf{x} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Homework. Due at the beginning of class on Wed, Sept 16.

In each of these, find m, n, r, and find the general solution, if there are any solutions. Check for consistency with Theorem 5.

1. pg. 26, # 14
2. pg. 26, # 20
3-5. pg. 42, # 6,8,12