Math 1180

Notes 11

1 Basis and dimension

Read: Chapter 4, through page 347

Definition: Let V be a vector space. **A set of vectors** $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ in V is called a "basis" for V if

$$(i) \quad \text{It is a linearly independent set}$$
$$(ii) \quad \text{span}\,\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\} = V.$$

Example: The set above $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is a basis for R^2. We showed that it was linearly independent, and spanning is easy, since for any (x, y),

$$\begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \qquad (1)$$

Here's another example of a basis. Consider the set

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

in R^3. I first show that this set is linearly independent.

Suppose that

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then

$$c_1 + c_3 = 0$$
$$c_1 + c_2 = 0$$
$$c_2 + c_3 = 0.$$

The corresponding matrix has the three vectors of our set as columns. We reduce it to echelon form:

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Since it has three pivots, and is 3×3, it is nonsingular, and so

$$A\mathbf{c} = \mathbf{0}$$

has only the trivial solution. Hence B is a linearly independent set.

I now show that $\text{span}\{B\} = R^3$. To show this we must show that we can express an arbitrary vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ in R^3 as a linear combination of the vectors in B. This means that there are c_1, c_2, c_3 such that

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

This means that we must show that the equation

$$A\mathbf{c} = \mathbf{b}$$

where $\mathbf{b}$ is the vector (x_1, x_2, x_3), can be solved for $\mathbf{c}$. (Note the different notation we are using from earlier.) But this follows in the same way that we showed that B is a linearly independent set. The 3×3 system $A\mathbf{c} = \mathbf{0}$ has only the zero solution. We have seen that this implies that $A\mathbf{c} = \mathbf{b}$ has a solution for any vector $\mathbf{b}$. (This is only true for square systems, but that is what we have here.)

We have therefore shown that B is a linearly independent set, and it spans R^3, and so it is a basis for R^3.

Next we give an example of a basis of a subspace. We consider the subspace V of R^4 consisting of all solutions of

$$x_1 + x_2 + x_3 + x_4 = 0. \qquad (2)$$

We know that this is a hyperplane, and it is apparent that there are three free variables, which we can choose to be x_2, x_3, x_4. We write

$$x_1 = -x_2 - x_3 - x_4.$$

Choosing each free variable, in turn, to be 1 with the others zero, we are led to the general solution

$$\mathbf{x} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3}$$

So every solution is a linear combination of the three vectors shown. Hence, the subspace V is spanned by these vectors. So we let

$$B_1 = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

This set of vectors is obviously linearly independent, since by looking at the last three components of the vectors we can see that none is a linear combination of the others. So we immediately see that B_1 is linearly independent and spans V. Hence, B_1 is a basis for the subspace V.

Dimension.

Note that in the preceding pages we gave two different bases for R^3. That is, we gave two different sets, each of which is a basis for R^3. Before we can define dimension, we have to answer the following question: Can there be two different bases for a subspace V of R^n which have different numbers of elements? The answer is no, and this is important enough to state as a Theorem.

Theorem 1 *(Theorem 2, page 339) If $\{\mathbf{v}_1, ..., v_k\}$ is a basis for V and $\{\mathbf{w}_1, ..., \mathbf{w}_m\}$ is also a basis for V, then $k = m$.*

Proof: I will give the same proof as the text, but write it down slightly differently.

If $k \neq m$, then one of them must be larger. Let's assume $m > k$.. Otherwise we switch the names of $\mathbf{v}$ and $\mathbf{w}$.

We will show that if $m > k$ then $\{\mathbf{w}_1, ..., \mathbf{w}_m\}$ is a linearly dependent set, which will contradict the definition of a basis.

To show that $\{\mathbf{w}_1, ..., \mathbf{w}_m\}$ is a linearly independent set, we must find a nonzero set of numbers $c_1, ..., c_m$ such that

$$c_1\mathbf{w}_1 + \cdots + c_m\mathbf{w}_m = \mathbf{0}. \tag{4}$$

Since $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is a basis, each $\mathbf{w}_i$ is a linear combination of $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$. Therefore there are numbers a_{ij} such that

$$\mathbf{w}_i = a_{1i}\mathbf{v}_1 + \cdots + a_{ki}\mathbf{v}_k.$$

We substitute this into (4) to get a rather complicated looking set of equations:

$$\begin{aligned} c_1 (a_{11}\mathbf{v}_1 + \cdots + a_{k1}\mathbf{v}_k) + c_2 (a_{12}\mathbf{v}_1 + \cdots + a_{k2}\mathbf{v}_k) \\ + \cdots + c_m (a_{1m}\mathbf{v}_1 + \cdots + a_{km}\mathbf{v}_k) = \mathbf{0}. \end{aligned}$$

Using summation signs:

$$\sum_{\iota=1}^{n} c_i \left(\sum_{j=1}^{k} a_{ji}\mathbf{v}_j \right) = \mathbf{0}.$$

We can reverse the order of any two finite summations. This gives

$$\sum_{j=1}^{k} \left(\sum_{\iota=1}^{n} c_i a_{ji} \right) \mathbf{v}_j = \mathbf{0}.$$

But $\mathbf{v}_1, ..., \mathbf{v}_k$ are linearly independent. This means that the coefficient of each $\mathbf{v}_j$ must be zero. This gives a set of k homogeneous equations in m unknowns, namely

$$\begin{aligned} c_1a_{11} + c_2a_{12} + \cdots + c_ma_{1m} &= 0 \\ &\cdot\cdot \\ c_1a_{k1} + c_2a_{k2} + \cdots + c_ma_{km} &= 0. \end{aligned}$$

We want to find a non-zero set of c_i. But since $m > k$, there are more unknowns than equations, so we know that there are non-zero solutions. This proves that $\{\mathbf{w}_1, ..., \mathbf{w}_m\}$ could not be a basis, a contradiction.

With this theorem, we can define dimension.

Definition: *The dimension of a subspace V is the number of vectors which are in any basis of V.*

From earlier, we have two examples for the space R^3, namely

$$B_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$
$$B_2 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

As a further example, we can consider the vector space P^2. The set of vectors $\{1, x, x^2\}$ clearly spans P^2. To see if it is a basis, we have to see if it is a linearly independent set. Suppose that

$$c_1 + c_2 x + c_3 x^2 = 0.$$

We have to think carefully about what this means. It means that this expression is the zero polynomial. It is zero for every x. So we can substitute different values of x in there. Substitute $x = 0, x = 1, x = 2$. We get

$$\begin{aligned} c_1 &= 0 \\ c_1 + c_2 + c_3 &= 0 \\ c_1 + 2c_2 + 4c_3 &= 0. \end{aligned}$$

We can use "forward substitution to get the answer:

$$\begin{aligned} c_1 &= 0 \\ c_2 &= -c_3 \\ 2(-c_3) + 4c_3 &= 0 \end{aligned}$$

so $c_3 = 0$ and hence $c_2 = 0$. This shows that $\{1, x, x^2\}$ is linearly independent. Hence, it is a basis for P^2 , and the dimension of P^2 is three. It is pretty easy to show that the set

$$\{p_1(x), p_2(x), p_3(x)\} = \left\{1, x, x^2 - \frac{1}{3}\right\}$$

is also a basis for P^2. This basis has a property that is (a) strange, (b) interesting, and (c) important:

$$\int_{-1}^{1} p_i(x)\, p_j(x)\, dx = 0$$

if $i \neq j$.

Question: What is the dimension of the set of solutions of $A\mathbf{x} = \mathbf{0}$ in R^4 if

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 0 & -1 & 2 \end{pmatrix}.$$

Proof: Find the general solution by putting A in echelon form:

$$A \to \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -6 & -4 & 2 \end{pmatrix}.$$

There are two pivots and two free variables, x_3 and x_4. We get:

$$x_2 = -\frac{2}{3}x_3 + \frac{1}{3}x_4$$
$$x_1 = -x_3 - 2x_2 = \frac{1}{3}x_3 - \frac{2}{3}x_4.$$

Thus,

$$\mathbf{x} = \mathbf{x}_3 \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ 0 \\ 1 \end{pmatrix}.$$

This means that the solution space is

$$span \left\{ \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ 0 \\ 1 \end{pmatrix} \right\}$$

From the last two components we see that this is a linearly independent set. Therefore it is a basis for the subspace of solutions, and this subspace has dimension 2.

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2 Fundamental subspaces for an $m \times n$ matrix.

Definition 2 *If A is an $m \times n$ matrix then*

$$N(A) = \{\mathbf{x} \in \mathbf{R}^n \mid A\mathbf{x} = \mathbf{0}\}$$

is called the "null space" of A.

We gave an example of this just above, where

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 0 & -1 & 2 \end{pmatrix}.$$

We saw that the null space is a two dimensional subspace of R^4. The dimension is found by using Gaussian elimination to see how many free variables there are.

Definition 3 *The "nullity" of A is the dimension of $N(A)$.*

Theorem 4 *The nullity of A is $n - r$, where r is the rank.*

This easily follows from our earlier studies of Gaussian elimination. The nullity is equal to the number of free variables in the solution to $A\mathbf{x} = \mathbf{0}$, which is $n - r$, r being the rank, which is equal to the number of pivots.

Here is a second space associated with a matrix A.

Definition 5 *The "column space" of A is span$\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ where $\mathbf{v}_1, .., \mathbf{v}_n$ are the columns of the $m \times n$ matrix A.*

So for the matrix A above, the column space is

$$c(A) = span\left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}\right\}.$$

We see that $C(A)$ is a subspace of R^m (R^2 in this case) .But a little thought shows that we don't need to use all of these vectors to get $c(A)$. We could use just the second and fourth vectors, because these two by themselves span all of R^2. Thus, a **basis** for $C(A)$ is

$$B = \left\{\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}\right\}.$$

But we could just as well use

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

because this spans the same space, which is all of R^2. Thus, $c(A) = R^2$, and its dimension is 2, which we saw earlier is the rank of A.

One more example, which is very easy to analyze.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We can see right away that A has rank 2. Also, the first two columns span the column space:

$$c(A) = span \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

The dimension of $C(A)$ is two, which is again the rank. And

$$N(A) = span \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

with dimension $2 = n - r$.

The columns space is also called the "range" of A, because in a sense it resembles the range of a function. The function in this case is

$$f(\mathbf{x}) = A\mathbf{x},$$

where $\mathbf{x}$ is in R^n. To see why the range of this function is $c(A)$, we recall the following fact:

If A is an $m \times n$ matrix, and $x \in R^n$, then Ax is a linear combination of the columns of A.

Homework:

1. pg. 331, # 38. Write all vectors as column vectors, including the given one, $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. In (b) find the simplest examples you can. In (c) use the definition

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} bf - ce \\ dc - fa \\ ae - bd \end{pmatrix}$$

2. pg. 331, # 48

3. pg. 345, # #28. Do this for $n = 3$.

4. Find bases for the range (column space) and null space for the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 0 \\ 1 & 4 & 8 & 12 \end{pmatrix}.$$

5. pg. 362, # 36. $\rho(A)$ is the rank of A. One relevant fact is that if $B\mathbf{x} = \mathbf{0}$, then $AB\mathbf{x} = \mathbf{0}$.