Math 1180, Linear Algebra, Notes #1
Aug. 31, 2009

Much of linear algebra is the study of solving systems of linear equations. Here is a sample system:

$$\begin{aligned} 2x_1 - x_2 \qquad &= 1 \\ -x_1 + 2x_2 - x_3 &= 0 \\ -x_2 + 2x_3 &= 0 \end{aligned}$$

You probably learned to solve such a system in school. From the first equation we get

$$x_2 = 2x_1 - 1$$

and from the second equation we get

$$\begin{aligned} x_3 &= -x_1 + 2x_2 \\ &= -x_1 + 2(2x_1 - 1) \\ &= 3x_1 - 2. \end{aligned}$$

Substituting into the third equation we get

$$-(2x_1 - 1) + 2(3x_1 - 2) = 0$$

or

$$4x_1 - 3 = 0$$

from which we get $x_1 = \frac{3}{4}$. Then we find that $x_2 = 2x_1 - 1 = \frac{1}{2}$ and $x_3 = 3x_1 - 2 = \frac{1}{4}$.

Ok, here's another system:

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ -x_{i-1} + 2x_i - x_{i+1} &= 0 \text{ for } i = 2, ..., 999999, \\ -x_{999,999} + 2x_{1,000,000} &= 0. \end{aligned}$$

Hmm. This is why we study linear algebra. Systems with a million or more equations are common in applications. Some of these can be solved on a computer, but on the other hand, it's easy to give a system with 100 equations which can't be solved on any current computer.

To give this example, first consider the following system:

$$\begin{aligned}
x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 + \frac{1}{4}x_4 + \frac{1}{5}x_5 &= -1 \\
\frac{1}{2}x_1 + \frac{1}{3}x_2 + \frac{1}{4}x_3 + \frac{1}{5}x_4 + \frac{1}{6}x_5 &= 1 \\
\frac{1}{3}x_2 + \frac{1}{4}x_2 + \frac{1}{5}x_3 + \frac{1}{6}x_4 + \frac{1}{7}x_5 &= -1 \\
\frac{1}{4}x_1 + \frac{1}{5}x_2 + \frac{1}{6}x_3 + \frac{1}{7}x_4 + \frac{1}{8}x_5 &= 1 \\
\frac{1}{5}x_1 + \frac{1}{6}x_2 + \frac{1}{7}x_3 + \frac{1}{8}x_4 + \frac{1}{9}x_5 &= -1
\end{aligned}$$

This is obviously too hard to do by hand. We will use a computer program. The best program for linear algebra is Matlab, which all engineering students will have heard of, and perhaps used. Here is the solution I get from Matlab, using "exact arithmetic".:

$$\begin{aligned}
x_1 &= -3405 \\
x_2 &= 63480 \\
x_3 &= -273630 \\
x_4 &= 413280 \\
x_5 &= -202230
\end{aligned}$$

The appearance of such large numbers in the answer, when the matrix is only 5×5 and there are no very large or very small numbers in the equations, may give us pause, as well it should. It turns out that if we use the same pattern to generate 100 equations with 100 unknowns, then no current computer can find the solution! I am confident that no conceivable computer could find the answer for 1000 equations in 1000 unknowns. My guess is that the storage required would be more than 10^{900} times the number of atoms in the universe, and the time required would far exceed any projected life span of the universe. (Yet, mathematicians, with pencil and paper, have found a formula which works. Unfortunately, changing the numbers by just a little, say replacing just one coefficient, can probably give a system for which no solution is known or can be found.) Fortunately, many systems which arise in applications can be solved, usually on a computer.

As the text points out, linear algebra can be considered from the point of view of geometry. To picture this geometry we need to stick to two or three dimensions, but once we understand these well, we will have better intuition about higher dimensional examples as well.

To see why geometry is involved, suppose we have two equations in two unknowns, such as

$$\begin{aligned} x - y &= 1 \\ x + y &= 2. \end{aligned}$$

(When there are only two equations, we will often use x, y instead of x_1, x_2.) Recall that we can graph each of these equations in the x, y plane, and each is a line. Any point (x, y) which satisfies these equations must lie on each of these lines. We can easily solve each equation for y and see that the lines are not parallel, so they intersect in only one point, which gives the solution. By adding and subtracting the equations we quickly see that $x = \frac{3}{2}, y = \frac{1}{2}$.

On the other hand, consider the equations

$$\begin{aligned} x - y &= 1 \\ x - y &= 2. \end{aligned}$$

Each of these equations is the equation of a line in R^2 (the x, y plane). But these lines do not intersect. It is obvious that there is no point (x, y) in the plane which satisfies both of these equations, since they are "inconsistent". Hence this set of equations has no solutions.

Finally, consider the equations

$$\begin{aligned} x - y &= 1 \\ 3x - 3y &= 3. \end{aligned}$$

These obviously are equations of the same line, and so there is an infinite number of solutions.

In much of the course we will discuss generalizations of these ideas. We will consider more equations and more unknowns. For example, we could have a set of two equations in three unknowns, such as

$$\begin{aligned} x - y + z &= 1 \\ x + y - 2z &= -2 \end{aligned} .$$

Can we think of these geometrically? Yes, if we recall three dimensional geometry from Calc III. We will discuss this in class.