The Riemann Integral

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1 Upper and lower sums

A *partition* of a closed interval $[a,b]$ is a subset $\mathcal{P}=\{x_0,x_1,\dots,x_n\}$ of $[a,b]$ with $a=x_0<x_1<\cdots<x_n=b$. A partition divides the interval $[a,b]$ into *subintervals* $I_j=[x_{j-1},x_j]$, for $j=1,\dots,n$. We'll call I_j the jth subinterval for the partition.

Now let f be a bounded function on the interval $[a,b]$, and let $\mathcal{P}=\{x_0,\dots,x_n\}$ be a partition of $[a,b]$. Let

$$m_j=\inf\{f(x):x\in I_j\},\qquad M_j=\sup\{f(x):x\in I_j\}$$

and define the upper and lower sums for the function f with respect to the partition $\mathcal{P}$ by

$$L(f;\mathcal{P})=\sum_{j=1}^{n}m_j(x_j-x_{j-1})$$

and

$$U(f;\mathcal{P})=\sum_{j=1}^{n}M_j(x_j-x_{j-1}).$$

Since $m_j\le M_j$ for every j, it follows that

$$L(f;\mathcal{P})\le U(f;\mathcal{P}).$$

We next ask how the upper and lower sums will be affected if we add a single new point x' to a the partition $\mathcal{P}$, to obtain a new partition $\mathcal{P}'$. Since x' is not equal to any of the original partition points, it's in the interior of one of the subintervals, say

$$x_{j'-1}<x'<x_{j'}.$$

Now most of the terms in the new upper and lower sums for the partition $\mathcal{P}'$ are the same as the corresponding terms in the original sums. The only difference is that the term corresponding to the subinterval $I_{j'}$ is split into two terms for the subintervals $[x_{j'-1}, x']$ and $[x', x_{j'}]$. Thus, in the upper sum, the term

$$M_{j'}(x_{j'} - x_{j'-1})$$

gets replaced by the two terms

$$M'(x' - x_{j'-1}) + M''(x_{j'} - x')$$

where

$$M' = \sup\{f(x) : x \in [x_{j'-1}, x']\}, \qquad M'' = \sup\{f(x) : x \in [x', x_{j'}]\}.$$

Notice that M' and M'' are both less than or equal to $M_{j'}$, so

$$M'(x'-x_{j'-1})+M''(x_{j'}-x') \leq M_{j'}(x'-x_{j'-1})+M_{j'}(x_{j'}-x') = M_{j'}(x_{j'}-x_{j'-1}).$$

Therefore, we have shown that

$$U(f;\mathcal{P}') \leq U(f;\mathcal{P}),$$

and similar reasoning for the lower sums gives

$$L(f;\mathcal{P}') \geq L(f;\mathcal{P}).$$

We have proved

Lemma 1.1. *If the partition $\mathcal{P}'$ is obtained from $\mathcal{P}$ by adding a single point, then*

$$L(f;\mathcal{P}) \leq L(f;\mathcal{P}') \leq U(f;\mathcal{P}') \leq U(f;\mathcal{P}).$$

We next ask what happens to the upper and lower sums if you add many points to a given partition. Let's call a partition $\mathcal{P}'$ a *refinement* of the partition $\mathcal{P}$ if $\mathcal{P} \subset \mathcal{P}'$. In this case, either $\mathcal{P}' = \mathcal{P}$, or you can get from $\mathcal{P}$ to $\mathcal{P}'$ in several steps, adding one point at each step. By Lemma 1.1, each time you add a point, the lower sums get bigger, and the upper sums get smaller. We've proved

Theorem 1.2. *Let f be any bounded function on $[a, b]$ and let $\mathcal{P}$ and $\mathcal{P}'$ be partitions of $[a, b]$ such that $\mathcal{P}'$ is a refinement of $\mathcal{P}$. Then*

$$L(f;\mathcal{P}) \leq L(f;\mathcal{P}') \leq U(f;\mathcal{P}') \leq U(f;\mathcal{P}).$$

Theorem 1.2 gives us a way of comparing upper and lower sums for partitions that are completely independent of each other. Let $\mathcal{P}$ and $\mathcal{Q}$ be *any* two partitions of $[a,b]$. Then the partition $\mathcal{P} \cup \mathcal{Q}$ is a refinement of both $\mathcal{P}$ and $\mathcal{Q}$, so Theorem 1.2 gives

$$L(f;\mathcal{P}) \leq L(f;\mathcal{P} \cup \mathcal{Q}) \leq U(f;\mathcal{P} \cup \mathcal{Q}) \leq U(f;\mathcal{Q}).$$

We therefore have

Corollary 1.3. *For any bounded function f on $[a,b]$ and any partitions $\mathcal{P}$ and $\mathcal{Q}$ of $[a,b]$ we have*

$$L(f;\mathcal{P}) \leq U(f;\mathcal{Q})$$

2 Integrability

A consequence of Corollary 1.3 is that the set of all lower sums is bounded above, while the set of all upper sums is bounded below. We define the *lower and upper integrals* of a bounded function f by

$$\begin{aligned} L\int_a^b f &= \sup\{L(f;\mathcal{P})\} \\ U\int_a^b f &= \inf\{U(f;\mathcal{P})\} \end{aligned}$$

where the supremum and infimum are taken over all partitions $\mathcal{P}$ of $[a,b]$. Corollary 1.3 ensures that both the lower and upper integrals are finite, and that

$$L\int_a^b f \leq U\int_a^b f.$$

It's possible that the above inequality is strict. In this case, we say that the function f is not integrable. We'll say f is *integrable* if the upper and lower integrals are equal, and it this case, we define $\int_a^b f$ to be the common value of the upper and lower integrals.

Example 2.1 (Constant functions)**.** *Let f have the constant value c on the interval $[a,b]$. Then for any partition $\mathcal{P}$, we have*

$$m_j = M_j = c$$

on each subinterval, so

$$\begin{aligned} L(f;\mathcal{P}) &= \sum_{j=1}^{n} m_j(x_j - x_{j-1}) \\ &= \sum_{j=1}^{n} c(x_j - x_{j-1}) \\ &= c\sum_{j=1}^{n}(x_j - x_{j-1}) \\ &= c(x_n - x_0) \\ &= c(b-a) \end{aligned}$$

so

$$L\int_a^b f = c(b-a),$$

and similarly

$$U\int_a^b f = c(b-a).$$

Therefore, f is integrable, and

$$\int_a^b f = c(b-a).$$

Example 2.2 (A non-integrable function)**.** *Define f on $[0,1]$ by $f(x)=0$ if x is rational, and $f(x)=1$ if x is irrational. Let $\mathcal{P}$ be any partition of $[0,1]$. Since the jth subinterval contains both rational and irrational numbers, we have $m_j = 0$ and $M_j = 1$. Therefore*

$$L(f;\mathcal{P}) = \sum_{j=1}^{n} 0(x_j - x_{j-1}) = 0$$

and

$$U(f;\mathcal{P}) = \sum_{j=1}^{n} 1(x_j - x_{j-1}) = 1$$

so the lower integral is 0 and the upper integral is 1. Therefore f is not integrable

3 A Cauchy criterion for integrability

Theorem 3.1 (Cauchy Criterion)**.** *Let f be a bounded function on $[a, b]$. Then f is integrable if and only if for every $\varepsilon > 0$ there is a partition $\mathcal{P}$ such that*

$$U(f; \mathcal{P}) - L(f; \mathcal{P}) \leq \varepsilon.$$

Proof. Suppose f is integrable. Let $\varepsilon > 0$ be given. By definition of integrability, we can approximate the integral to within $\frac{\varepsilon}{2}$ from above and below by upper and lower sums respectively. Thus, there are partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ such that

$$U(f; \mathcal{P}_1) - \int_a^b f < \frac{\varepsilon}{2}$$

and

$$\int_a^b f - L(f; \mathcal{P}_2) < \frac{\varepsilon}{2}.$$

Adding these two inequalities gives

$$U(f; \mathcal{P}_1) - L(f; \mathcal{P}_2) < \varepsilon.$$

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then $\mathcal{P}$ is a refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$, so Theorem 1.2 gives

$$L(f; \mathcal{P}_1) \leq L(f; \mathcal{P}) \leq U(f; \mathcal{P}) \leq U(f; \mathcal{P}_2)$$

and so

$$U(f; \mathcal{P}) - L(f; \mathcal{P}) \leq U(f; \mathcal{P}_1) - L(f; \mathcal{P}_2) < \varepsilon.$$

For the converse, let $\varepsilon > 0$ be arbitrary, and let $\mathcal{P}$ be as in the statement of the theorem. Then

$$U\!\int_a^b f \leq U(f; \mathcal{P}) \leq L(f; \mathcal{P}) + \varepsilon \leq L\!\int_a^b f + \varepsilon$$

and since $\varepsilon > 0$ is arbitrary,

$$U\!\int_a^b f \leq L\!\int_a^b f.$$

Since the reverse inequality always holds, the upper and lower integrals are equal, so f is integrable. ☐

4 Integrability of monotone functions

Theorem 4.1. *If f is monotone on $[a, b]$, then f is integrable on $[a, b]$.*

Proof. We'll prove the theorem under the assumption that f is increasing. The proof for decreasing functions is similar.

For any positive integer n, let $\mathcal{P}_n$ be the partition that divides $[a, b]$ into n subintervals of equal length $\frac{b-a}{n}$. To be explicit, the partition points are

$$x_j = a + j\frac{b-a}{n}.$$

Since f is increasing, the infimum and supremum of f over the jth subinterval $[x_{j-1}, x_j]$ are

$$m_j = f(x_{j-1}), \qquad M_j = f(x_j).$$

The upper sum is

$$U(f; \mathcal{P}_n) = \sum_{j=1}^{n} M_j(x_j - x_{j-1}) = \frac{b-a}{n}\sum_{j=1}^{n} f(x_j)$$

and the lower sum is

$$L(f; \mathcal{P}_n) = \frac{b-a}{n}\sum_{j=1}^{n} f(x_{j-1})$$

so

$$\begin{aligned} U(f; \mathcal{P}_n) - L(f; \mathcal{P}_n) &= \frac{b-a}{n}\sum_{j=1}^{n}(f(x_j) - f(x_{j-1})) \\ &= \frac{b-a}{n}(f(x_n) - f(x_0)) \\ &= \frac{b-a}{n}(f(b) - f(a)). \end{aligned}$$

Now let $\varepsilon > 0$ be arbitrary. Choose n sufficiently large that $\frac{(b-a)(f(b)-f(a))}{n} < \varepsilon$. Then

$$U(f; \mathcal{P}_n) - L(f; \mathcal{P}_n) < \varepsilon.$$

By the Cauchy criterion for integrability, f is integrable. $\square$

Example 4.2. *We will evaluate the integral* $\int_0^1 x$*. Note first that, since the function* $f(x) = x$ *is monotone, integrability is guaranteed by the previous theorem.*

Fix a natural number n*, and let* $\mathcal{P}_n$ *be the partition of* $[0,1]$ *with partition points* $x_j = j/n$*. Then*

$$\begin{aligned} U(f;\mathcal{P}_n) &= \sum_{j=1}^{n} x_j(x_j - x_{j-1}) \\ &= \sum_{j=1}^{n} \frac{j}{n}\frac{1}{n} \\ &= \frac{1}{n^2}\sum_{j=1}^{n} j \\ &= \frac{n(n+1)}{2n^2} \\ &= \frac{1}{2} + \frac{1}{2n} \end{aligned}$$

and similarly

$$L(f;\mathcal{P}_n) = \frac{1}{2} - \frac{1}{2n}.$$

Since the integral is greater than or equal to every lower sum and less than or equal to every upper sum, we have

$$\frac{1}{2} - \frac{1}{2n} \le \int_0^1 x \le \frac{1}{2} + \frac{1}{2n}.$$

Since this holds for every positive integer n*, it follows that*

$$\int_0^1 x = \frac{1}{2}.$$

5 Integrability of continuous functions

Theorem 5.1. *If* f *is continuous on* $[a,b]$*, then* f *is integrable on* $[a,b]$*.*

Proof. First, the Boundedness Theorem ensures that f is bounded, so we can check for integrability using the Cauchy Criterion. Let $\varepsilon > 0$ be arbitrary.

By the Uniform Continuity Theorem, f is uniformly continuous, so there is a $\delta > 0$ such that

$$|f(x) - f(x')| < \frac{\varepsilon}{b-a}$$

for every $x, x' \in [a, b]$ with $|x - x'| < \delta$. Let $\mathcal{P}$ be any partition such that each subinterval has length less than δ. Since each subinterval I_j is closed, the Max Min Theorem ensures that the restriction of f to I_j achieves a maximum and minimum value, so $m_j = f(s_j)$ and $M_j = f(t_j)$ for some $s_j, t_j \in I_j$, and since I_j has length less then δ, we have $|t_j - s_j| < \delta$. By the way we chose δ,

$$M_j - m_j = f(t_j) - f(s_j) < \frac{\varepsilon}{b-a}$$

so

$$\begin{aligned} U(f;\mathcal{P}) - L(f;\mathcal{P}) &= \sum M_j(x_j - x_{j-1}) - \sum m_j(x_j - x_{j-1}) \\ &= \sum (M_j - m_j)(x_j - x_{j-1}) < \sum \frac{\varepsilon}{b-a}(x_j - x_{j-1}) \\ &= \frac{\varepsilon}{b-a} \sum (x_j - x_{j-1}) = \frac{\varepsilon}{b-a}(b-a) = \varepsilon \end{aligned}$$

By the Cauchy Criterion, f is integrable. □

6 Properties of integrals

Theorem 6.1 (Linearity). *1. If f and g are integrable on $[a, b]$, then $f+g$ is integrable, and*

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

2. If f is integrable on $[a, b]$ and $c \in \mathbb{R}$, then cf is integrable and

$$\int_a^b cf = c \int_a^b f.$$

Proof. We'll only prove the first assertion. The second is similar, but slightly easier, and will be left as an exercise.

Let $\varepsilon > 0$, and choose partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ such that

$$L(f;\mathcal{P}_1) > \int_a^b f - \frac{\varepsilon}{2}$$

and

$$L(g;\mathcal{P}_2) > \int_a^b g - \frac{\varepsilon}{2}.$$

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Since, by Theorem 1.2, passing to a refinement will increase the lower sum, the above inequalities continue to hold when $\mathcal{P}_1$ and $\mathcal{P}_2$ are replaced by $\mathcal{P}$. Therefore

$$\begin{aligned} L\!\int_a^b (f+g) &\geq L(f+g;\mathcal{P}) \geq L(f;\mathcal{P}) + L(g;\mathcal{P}) \\ &\geq L(f;\mathcal{P}_1) + L(g;\mathcal{P}_2) \\ &> \int_a^b f + \int_a^b g - \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary,

$$L\!\int_a^b (f+g) \geq \int_a^b f + \int_a^b g$$

A similar argument gives

$$U\!\int_a^b (f+g) \leq \int_a^b f + \int_a^b g$$

so it follows that

$$L\!\int_a^b (f+g) = U\!\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

□

Theorem 6.2 (Monotonicity)**.** *If f and g are integrable on $[a,b]$, and if $f(x) \leq g(x)$ for every $x \in [a,b]$, then*

$$\int_a^b f \leq \int_a^b g.$$

The proof is elementary, and is left as an exercise.

Theorem 6.3 (Additivity)**.** *Let f be integrable on $[a,b]$, and let $a < c < b$. Then f is integrable on $[a,c]$ and $[c,b]$, and*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof. Let $\varepsilon > 0$. Since f is integrable on $[a, b]$, there is a partition $\mathcal{P}$ of $[a, b]$ such that

$$U(f; \mathcal{P}) < U\!\int_a^b f + \varepsilon.$$

By Lemma 1.1, we may assume that $\mathcal{P}$ contains c. (If not, adding it will make the left side even smaller.) Let $\mathcal{P}'$ and $\mathcal{P}''$ be the partitions of $[a, c]$ and $[c, b]$ obtained by intersecting $\mathcal{P}$ with $[a, c]$ and $[c, b]$ respectively. Then

$$U\!\int_a^c f + U\!\int_c^b f \leq U(f; \mathcal{P}') + U(f; \mathcal{P}'') = U(f; \mathcal{P}) < \int_a^b f + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$U\!\int_a^c f + U\!\int_c^b f \leq \int_a^b f.$$

A similar argument using lower sums shows

$$\int_a^b f \leq L\!\int_a^c f + L\!\int_c^b f$$

so

$$U\!\int_a^c f + U\!\int_c^b f \leq \int_a^b f \leq L\!\int_a^c f + L\!\int_c^b f.$$

Since the right side is always less than or equal to the left side, we have equality throughout, and the result follows. ☐

Theorem 6.4 (Triangle Inequality)**.** *If f is integrable on $[a, b]$, then $|f|$ is also integrable, and*

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

Proof. It's easy to check that for any partition $\mathcal{P}$,

$$U(|f|; \mathcal{P}) - L(|f|; \mathcal{P}) \leq U(f; \mathcal{P}) - L(f; \mathcal{P})$$

so it follows from the Cauchy Criterion that $|f|$ is integrable. Also, since

$$-|f(x)| \leq f(x) \leq |f(x)|$$

for every x, monotonicity gives

$$-\int_a^b |f(x)| \leq \int_a^b f(x) \leq \int_a^b |f(x)|$$

or equivalently

$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

□

7 The Fundamental Theorem of Calculus

Theorem 7.1 (Fundamental Theorem of Calculus I)**.** *Let f be an integrable function on $[a,b]$, and let F be a continuous function on $[a,b]$ such that $F'(x) = f(x)$ for every $x \in (a,b)$. Then*

$$\int_a^b f = F(b) - F(a).$$

Proof. Let $\mathcal{P} = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$. Applying the Mean Value Theorem on each subinterval gives $c_j \in (x_{j-1}, x_j)$ such that

$$\begin{aligned} F(b) - F(a) &= \sum (F(x_j) - F(x_{j-1})) = \sum F'(c_j)(x_j - x_{j-1}) \\ &= \sum f(c_j)(x_j - x_{j-1}) \end{aligned}$$

Clearly

$$m_j \leq f(c_j) \leq M_j$$

where m_j and M_j are the infimum and supremum of $f(x)$ on the jth subinterval. Therefore

$$L(f;\mathcal{P}) \leq F(b) - F(a) \leq U(f;\mathcal{P}).$$

Since $\int_a^b f$ can be approximated to any prescribed accuracy by upper and lower sums, it follows that

$$\int_a^b f = F(b) - F(a).$$

□

Theorem 7.2 (Fundamental Theorem of Calculus II)**.** *Let f be integrable on $[a, b]$, and let*

$$F(x) = \int_a^x f.$$

Then F is Lipschitz continuous, and if f is continuous at x_0, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Proof. By additivity, we have

$$F(x) - F(x_0) = \int_{x_0}^x f.$$

Since f is integrable, it's bounded, say $|f| \leq M$. Therefore if $x_0 < x$, the Triangle Inequality and monotonicity give

$$|F(x) - F(x_0)| = \left| \int_{x_0}^x f \right| \leq \int_{x_0}^x |f| \leq \int_{x_0}^x M = M(x - x_0)$$

which establishes Lipschitz continuity.

If f is continuous at x_0, then for $x > x_0$

$$\frac{F(x) - F(x_0)}{x - x_0} = \frac{1}{x - x_0} \int_{x_0}^x f$$

so

$$\begin{aligned} \left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| &= \left| \frac{1}{x - x_0} \int_{x_0}^x f - \frac{1}{x - x_0} \int_{x_0}^x f(x_0) \right| \\ &= \left| \frac{1}{x - x_0} \int_{x_0}^x (f - f(x_0)) \right| \\ &\leq \frac{1}{x - x_0} \int_{x_0}^x |f - f(x_0)| \end{aligned}$$

Let $\varepsilon > 0$. Since f is continuous at x_0, there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$. Therefore, for $x_0 < x < x_0 + \delta$,

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| \leq \frac{1}{x - x_0} \int_{x_0}^x \varepsilon = \varepsilon$$

and therefore

$$\lim_{x \to x_0^+} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0).$$

We leave as an exercise that the left hand limit is also equal to $f(x_0)$. ☐