.Math 0450
Honors intro to analysis
Spring, 2009
Notes #9
one page addendum added

1 Section 3.1, definition of the limit of a sequence

1.0.1 read first: section 3.1

This is one of the most important sections in the course. To understand mathematical analysis, you have to understand how to show that a sequence converges.

I won't repeat the definition in the text (3.1.1).[1] Be sure to read it carefully. Here I will do several examples, perhaps using slightly different steps or language from the text.

Example 3.1.6(c), pg. 56. Show that $\lim\left(\frac{3n+2}{n+1}\right)=3$.

Proof. First observe that

$$\left|\frac{3n+2}{n+1}-3\right|=\left|\frac{(3n+2)-3(n+1)}{n+1}\right|=\left|\frac{-1}{n+1}\right|=\frac{1}{n+1}. \tag{1}$$

Also,

$$\frac{1}{n+1}<\frac{1}{n}.$$

Now suppose that $\varepsilon>0$. We have shown that if

$$\frac{1}{n}\leq\varepsilon,$$

then

$$\left|\frac{3n+2}{n+1}-3\right|<\varepsilon.$$

But

$$\frac{1}{n}\leq\varepsilon\Leftrightarrow n\geq\frac{1}{\varepsilon}.$$

[1] See the addendum at the end of these notes for an important comment about the definition in the text.

Hence, let $K(\varepsilon) = \frac{1}{\varepsilon}$. If $n \geq K(\varepsilon)$, then

$$\left| \frac{3n+2}{n+1} - 3 \right| = \frac{1}{n+1} < \frac{1}{n} \leq \varepsilon.$$

Therefore, $\lim \left(\frac{3n+2}{n+1} \right) = 3$. ■

Remark 1 *This proof is somewhat long, because I tried to motivate the choice of $K(\varepsilon)$ before giving its formula. Some people like this kind of proof, but others prefer a more logical proof which seems to pull $K(\varepsilon)$ out of the air. So here's a second proof:*

Proof. (Proof two) Suppose that $\varepsilon > 0$. Let $K(\varepsilon) = \frac{1}{\varepsilon}$. If $n \in R$ and $n \geq K(\varepsilon)$, then

$$\left| \frac{3n+2}{n+1} - 3 \right| = \left| \frac{(3n+2) - 3(n+1)}{n+1} \right| = \left| \frac{-1}{n+1} \right| = \frac{1}{n+1} < \frac{1}{n} \leq \varepsilon.$$

Hence the sequence $\left(\frac{3n+2}{n+1} \right)$ converges to 3. ■

A typical student response upon seeing this proof is: How did you find $K(\varepsilon)$? With more experience, you realize that the writer of the proof worked backward, as we did in the first proof, but then rewrote to make the proof as short as possible.

But the writer did not succeed:

Proof. (proof three). Suppose that $\varepsilon > 0$. Let $K(\varepsilon) = \frac{1}{\varepsilon}$. If $n \geq K(\varepsilon)$, then

$$\left| \frac{3n+2}{n+1} - 3 \right| = \frac{1}{n+1} < \frac{1}{n} \leq \varepsilon.$$

Hence $\left(\frac{3n+2}{n+1} \right)$ converges to 3. ■

I hesitate to say that this is now as short as possible. I could have written the third sentence as "If $n \geq K(\varepsilon)$, then $\left| \frac{3n+2}{n+1} - 3 \right| < \varepsilon$." There is always a question of just how much detail one puts in a proof. The algebraic steps which are omitted in the shorter versions are straightforward, and any mathematician could fill them in. Most mathematicians would say that it is sufficient to give the formula for $K(\varepsilon)$, and leave the calculation that it works up to the reader.

How far one goes in leaving calculations up to the reader is pretty subjective. Some readers feel mildly insulted if you think they need too many details. Also, there is a sense that when there are a lot of details, the elegance of the proof is masked. Others will feel relieved that they don't have to spend time working them out for themselves, since they can follow the more detailed steps in their head as they read along. But, to continue the argument, others would then say that if you don't work the steps out for yourself, there is too much tendency to skip over difficulties and not really understand what is going on. Many research mathematicians take that view. Many students do not.

Notice that in two of the proofs I did not bother to say that $n \in R$. The letter n is used in the statement of the problem, in the expression $\left(\frac{3n+2}{n+1}\right)$, and the parentheses indicate that this is a sequence. The word "sequence" implies a function with domain $\mathcal{N}$, and so it is not necessary to belabor this fact further. The second proof is overly pedantic at least at this point.

I could have said: Prove that $\lim_{n\to\infty} \frac{3n+2}{n+1} = 3$. Strictly speaking, as we will see later, this statement does not require that n be an integer. However custom strongly dictates that use of the letters i, j, k, m, or n means that you are talking about a sequence, not a function of a real number, as in $\frac{3x+2}{x+1}$. We will deal with limits of functions with domain R later. We will see that usually they are no more difficult.

pg. 60, exercise 6(a). Prove that $\lim\left(\frac{1}{\sqrt{n+7}}\right) = 0$.

Proof: Suppose that $\varepsilon > 0$. Let $K(\varepsilon) = \frac{1}{\varepsilon^2}$. If $n \geq K(\varepsilon)$, then

$$\left|\frac{1}{\sqrt{n+7}} - 0\right| = \frac{1}{\sqrt{n+7}} < \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{K(\varepsilon)}} = \varepsilon,$$

Hence, $\lim\left(\frac{1}{\sqrt{n+7}}\right) = 0$.

Remark 2 *I took the attitude that you could work backward in this proof to see where I got $K(\varepsilon)$.*

pg. 60, # 6(b) Prove that $\lim\left(\frac{2n}{n+2}\right) = 2$.

Proof: Suppose that $\varepsilon > 0$. Let $K(\varepsilon) = \frac{5}{\varepsilon}$. If $n \geq K(\varepsilon)$, then

$$\left|\frac{2n}{n+2} - 2\right| = \left|\frac{-4}{n+2}\right| < \frac{4}{n} < \frac{5}{n} \leq \frac{5}{K(\varepsilon)} = \varepsilon. \tag{2}$$

Hence, $\frac{2n}{n+2} \to 2$ as $n \to \infty$.

Remark 3 *In this proof I used a string of inequalities that began with* $\left|\frac{2n}{n+2}-2\right|$ *and ended with* ε. *Some of these were* $<$ *and some were* $\leq$. *If at least one in the middle is* $<$, *I have proved what I wanted to prove, namely that if* $n \geq K(\varepsilon)$, *then* $\left|\frac{2n}{n+2}-2\right| < \varepsilon$.

Remark 4 *In the definition of limit, it doesn't make any difference if we use* $n \geq K(\varepsilon)$ *or* $n > K(\varepsilon)$. *Can you see why? When I say "it doesn't make any difference", what do I mean?*

Here's a second proof for this one: Suppose that $\varepsilon > 0$. Let $K(\varepsilon) = \frac{500}{\varepsilon}$. If $n \geq K(\varepsilon)$, then

$$\left|\frac{2n}{n+2}-2\right| = \left|\frac{-4}{n+2}\right| < \frac{4}{n} < \frac{500}{n} \leq \frac{5}{K(\varepsilon)} = \varepsilon. \tag{3}$$

Hence, $\frac{2n}{n+2} \to 2$ as $n \to \infty$.

Remark 5 *This proof makes the point that you don't have to find the "best possible"* $K(\varepsilon)$. *If your* $K(\varepsilon)$ *is larger than it has to be, so what? You have fulfilled the definition, which requires you to find* ***some*** $K(\varepsilon)$, *not necessarily the smallest. In fact, our first proof used a larger* $K(\varepsilon)$ *than was necessary. We could have said: Let* $K(\varepsilon) = \frac{4}{\varepsilon}$. *If* $n \geq K(\varepsilon)$, *then*

$$\left|\frac{2n}{n+2}-2\right| = \left|\frac{-4}{n+2}\right| < \frac{4}{n} \leq \frac{4}{K(\varepsilon)} = \varepsilon.$$

There is one "strict" inequality in the string, so it works.

Homework, due Feb. 3
pg. 43, # 10
pg. 59 # 5(d). 6(c), 6(d), 8, 10 (used often), 16 (Several of these have hints. You need to prove the hint, not just use it.

Addendum: In the definition in the text, $K(\varepsilon)$ is required to be a positive integer. This is not necessary, and in my opinion makes the wording of limit proofs a little more awkward than necessary. We will allow $K(\varepsilon)$ to be any positive real number. (It could even be negative, but in the context, this would be pointless; if you can find a negative $K(\varepsilon)$, you could always replace it with $K(\varepsilon) = 1$.)

The justification for this follows:

Proposition 6 *Suppose that* (x_n) *is a sequence,* $x \in R$*, and for each* $\varepsilon > 0$*, there is a* $K(\varepsilon) \in R$ *such that if* $n \geq K(\varepsilon)$*, then* $|x_n - x| < \varepsilon$*. Then for each* $\varepsilon > 0$ *there is a positive integer* $K_1(\varepsilon)$ *with the same property.*

Proof. Let $K_1(\varepsilon) = [[K(\varepsilon) + 1]]$, where $[[]]$ denotes the "greatest integer function". That is, for any $r \in R$, $[[r]]$ is the largest element of Z which is less than or equal to r. (Z is the set of all integers). If $n \geq K_1(\varepsilon)$, then $n \geq K(\varepsilon)$, and so $|x_n - x| < \varepsilon$. ■

This means that we can give formulas like $K(\varepsilon) = \frac{1}{\varepsilon}$, and not worry about whether this is an integer. The proposition is based on our earlier comment that there is no need to choose the "best possible" $K(\varepsilon)$.

(Do we need to prove the converse of this proposition?)