.Math 0450
Honors intro to analysis
Spring, 2009
Notes #8

1 Proof of the uncountability of R.

The text gives a proof based on the nested interval property (2.5.2, or on pg. 4 of notes 7). I will be a little more detailed.

Proof. It is sufficient to show that the interval $(0,1)$ is uncountable. Suppose that this interval is countable, and that $f : \mathcal{N} \to (0,1)$ is a bijection. Let $x_i = f(i)$. We obtain a contradiction by showing that there is an $x \in (0,1)$ with $x \neq x_i$ for all $i \in \mathcal{N}$. To do this we use induction to construct a nested set of closed intervals as follows:

If $x_1 < \frac{1}{2}$, let $I_1 = \left[\frac{1}{2}, \frac{3}{4}\right]$. If $x_1 \geq \frac{1}{2}$, let $I_1 = \left[\frac{1}{4}, \frac{1}{3}\right]$. Therefore, in either case, $x_1 \notin I_1$, I_1 is closed, and $I_1 \subset (0,1)$.

Now suppose that closed intervals $I_1, ..., I_k$ have been constructed so that (i) $x_i \notin I_i$ for $i = 1, ..., k$, and (ii) $I_{i+1} \subseteq I_i$ for $1 \leq i \leq k-1$. We now construct I_{k+1}. Let $I_k = [a_k, b_k]$. Let $\varepsilon_k = b_k - a_k$. If $x_{k+1} \leq \frac{a_k+b_k}{2} = a_k + \frac{1}{2}\varepsilon_k$, let $a_{k+1} = a_k + \frac{2}{3}\varepsilon_k$ and $b_{k+1} = b_k$. If $x_{k+1} > \frac{a_k+b_k}{2}$, let $a_{k+1} = a_k$ and $b_{k+1} = a_k + \frac{1}{3}\varepsilon_k$. In either case, $I_{k+1} \subseteq I_k \subset (0,1)$ and $x_{k+1} \notin I_{k+1}$. Since $\{I_n\}$ is a nested sequence of closed bounded intervals, there is an $x \in \cap_{n=1}^{\infty} I_n$, by the nested interval theorem. Further $x \in (0,1)$, and $x \neq x_i$ for each $i \in \mathcal{N}$. Hence, f is not a surjection and so not a bijection, which is a contradiction. This shows that $(0,1)$ is uncountable. ■

Remark 1 *This proof uses an extension of the principle of induction. We defined the sequence of intervals "inductively". It seems a little tricky to justify this using the Theorem about induction from section 1.2, and the authors do not try. Neither will I! We did not try to define $\mathcal{N}$, and it seems likely that this technique would be needed there also. It is used frequently in analysis and other parts of mathematics.*

Remark 2 *We will not cover the material on pages 48-50.*

Remarks on homework:

pg. 43, #8. Part (a) was assigned. Here is part (b). Notice the different notation I am using from that in the text.

$$g(y) = \inf_{0<x<1} 2x + y = y$$

$$\sup_{0<y<1} g(y) = \sup_{0<y<1} \left(\inf_{0<x<1} 2x + y \right) = \sup_{0<y<1} y = 1.$$

In your assignment, you should compare the answers to (a) and (b)

pg. 43, #9. Again, (a) was assigned. Here is (b):

$$g(y) = \inf_{0<x<1} h(x, y) = 0$$

$$\sup_{0<y<1} g(y) = \sup_{0<y<1} \left(\inf_{0<x<1} (h(x, y)) \right) = \sup_{0<y<1} 0 = 0.$$

Again, compare the answers to (a) and (b).