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## 1 Sections 2.4

### 1.1 read first: 2.4.

The last theorem in 2.4 is called the “density theorem”. The term “dense” is defined informally on page 42. just before the statement of the theorem. I don’t think we will need a more formal definition.

The proof of the theorem in the text refers back to some earlier theorems which I have not stated. Here I will try to give more of the motivation of the proof.

**Theorem 1** *If  $x \in R$  and  $y \in R$  and  $x < y$ , then there is a rational number  $r$  such that  $x < r < y$ .*

**Remark 2** *It doesn’t matter how close  $x$  and  $y$  are, or whether they are rational or irrational. There is always a rational number in between. It probably seems, not for the first time in this course, that this is something obvious. Between 1 and  $1 + \frac{1}{1000\sqrt{2}}$  there is the rational number 1.00000000001. But on the scale of “obvious” mathematical statements, is this more obvious than that  $\mathcal{N}$  is an unbounded set? It’s surprising that, first, either of these things should need to be proved, but they do<sup>1</sup>, and second, that they can both be proved based on the completeness axiom.*

Before giving the proof, here is a Corollary:

**Corollary 3** *If  $x \in R$  and  $y \in R$  and  $x < y$ , then there is an irrational number  $z$  with  $x < y < z$ .*

**Remark 4** *Again, it doesn’t matter how close  $x$  and  $y$  are, or whether they are rational or irrational.*

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<sup>1</sup>Without the completeness axiom, one can create examples of a set with two operations, say “ $\oplus$ ” and “ $\otimes$ ” which obey all of the previous axioms and yet are not equivalent to  $R$ , because they don’t have all of the properties of  $R$ . An example is given in the Wikipedia article “Archimedean property”, under the heading “Example of a non-Archimedean ordered field”.

**Proof.**

**Lemma 5** *If  $\varepsilon > 0$ , then there is an  $n \in \mathcal{N}$  with  $0 < \frac{1}{n} < \varepsilon$ .*

■

**Proof.** (proof of lemma). (We just about proved this when we were proving that  $\inf \{ \frac{1}{n} : n \in \mathcal{N} \} = 0$ .) Since  $\varepsilon > 0$ ,  $\frac{1}{\varepsilon} \in \mathcal{R}$ , and  $\frac{1}{\varepsilon} > 0$ . By the Archimedean property, there is an  $n$  such that  $n > \frac{1}{\varepsilon}$ . But then,  $\frac{1}{n} < \varepsilon$ , proving the lemma.

Continuing with the proof of the theorem, we can assume that  $x > 0$ , for if  $x < 0$  and  $y > 0$ , then we can choose a new  $x$ , say  $x_1$ , with

$$x < 0 < x_1 < y.$$

(The existence of  $x_1$  follows from the lemma, since we assumed  $y > 0$ .) Then we can find  $r$  with  $x_1 < r < y$ , which implies that  $x < r < y$ .) Also, if  $x < 0$  and  $y < 0$ , we can consider  $-x$  and  $-y$ .

So, we now have  $0 < x < y$ . Choose  $n \in \mathcal{N}$  so that  $\frac{1}{n} < \min \{x, y - x\}$ . This can be done by the Lemma. With  $n$  fixed, let

$$A = \left\{ \frac{k}{n} \in \mathcal{Q}^+ : \frac{k}{n} \leq x \right\}.$$

Then  $A$  contains at least the rational number  $\frac{1}{n}$ . Also,  $A$  is a finite set, because for large enough  $k$ ,  $\frac{k}{n} > x$ . (This follows from the Archimedean principle.) Hence, as we saw,  $\sup A \in A$ . Thus,  $\sup A = \frac{m}{n}$  for some unique  $m$  which depends on  $x$ . Further,  $\frac{m+1}{n} > x$ . However,  $\frac{m+1}{n} = \frac{m}{n} + \frac{1}{n} < x + \frac{1}{n} < y$  because  $\frac{1}{n} < y - x$ . (Recall how  $n$  was chosen.) Therefore,  $\frac{k+1}{n}$  is a rational number which lies between  $x$  and  $y$ . This proves the Theorem. ■

Here are some questions which will test your understanding of this proof: If you can't answer some of these, discuss with fellow students or ask in class or recitation.

1. Why could we assert at one stage that  $\frac{m+1}{n} > x$ ?
2. Explain in more detail why near the end we could state that  $\frac{1}{n} < y - x$ .
3. Explain in more detail how we know that  $A$  is a finite set.
4. What example in section 2.3 justifies the statement  $\sup A \in A$ ?
5. What is the set  $\mathcal{Q}^+$  referred to in the definition of  $A$ ?

6. What results in section 2.1 are needed to justify the claim in the proof of the Lemma that  $\frac{1}{\varepsilon} > 0$ ?

7. I justify several steps by the Lemma. For example, I claim that the existence of  $x_1$  follows from the lemma. There is no mention of any  $x_1$  or  $y$  in the Lemma. How is that claim justified?

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**Proof of Corollary: (Very neat, from the text):** Again we can assume that  $0 < x < y$ . From the Theorem it follows that there is a rational number  $r$  such that

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}.$$

But if  $r$  is rational, then  $\sqrt{2}r$  is irrational. (Otherwise,  $\sqrt{2} = \frac{\sqrt{2}r}{r}$  would be the quotient of two rational numbers and so rational.) Since

$$x < \sqrt{2}r < y,$$

the Corollary is proved.

## 2 Section 2.5.

### 2.0.1 read first: 2.5

We first define various sorts of intervals, such as an “open interval” and a “closed interval”. The set  $\{x \in R : 0 < x < 1\}$  is an open interval, while  $\{x \in R : 0 \leq x \leq 1\}$  is a closed interval. One contains its endpoints, the other does not. A general open interval can be denoted by  $(a, b)$ , where it is understood that  $a < b$ . A closed interval is usually written  $[a, b]$ . Sometimes we use “ $a = -\infty$ ” or “ $b = \infty$ ”, always with an “open bracket”, i.e.  $($ , not  $]$ . For example, we can consider the open interval  $(3, \infty)$ . This simply means that the interval is

$$\{x \in R : x > 3\}.$$

Notice that I don't have to use the symbol  $\infty$  to define this interval. This set is still called an interval. We also use  $(-\infty, \infty)$  to denote  $\{x \in R\}$ . In other words,  $(-\infty, \infty) = R$ .

The interval  $(-\infty, \infty)$  has (for technical reasons which I will not go into now), the property that it is both open and closed. Also,  $[0, \infty)$  and  $(-\infty, 0]$  are closed.

Theorem 2.5.1, the “Characterization theorem” is a rather strange theorem, the use of which is hard to envisage. When we need it, I will come back to it!

An important result in this section is the proof that  $R$  is uncountable. Instead of proving this directly about  $R$ , we prove it for simply the interval  $(0, 1)$ . Since this is a subset of  $R$ ,  $R$  must also be uncountable, because any subset of a countable set is also countable, by a previous theorem.

There are two proofs given. In class I will give the most famous one first: Cantor’s diagonal argument. While rigorous, I think it is still a little mysterious. The other proof, using nested intervals, I find more concrete.

The nested interval theorem is very useful. It is important that it requires using **closed** and bounded intervals. We will illustrate this below.

**Theorem 6** *Suppose that for each  $n \in \mathcal{N}$ ,  $I_n$  is a closed bounded interval, and furthermore that for each  $n \in \mathcal{N}$ ,  $I_{n+1} \subseteq I_n$ . Then there is some  $x \in R$  such that  $x \in I_n$  for every  $n$ .*

The conclusion of the theorem can be written:

$$x \in \bigcap_{n=1}^{\infty} I_n$$

The proof is in the text. Here are some examples:

**Example 7**

$$I_n = \left[ 0, \frac{1}{n} \right].$$

Then

$$0 \in \bigcap_{n=1}^{\infty} I_n$$

**Example 8**

$$I_n = \left[ -\frac{1}{n}, \frac{1}{n} \right].$$

Again,

$$0 \in \bigcap_{n=1}^{\infty} I_n$$

**Example 9**

$$I_n = \left(0, \frac{1}{n}\right).$$

Notice that these are nested intervals, However,

$$\bigcap_{n=1}^{\infty} I_n = \phi.$$

It is worth explaining this further. To prove that  $\bigcap_{n=1}^{\infty} I_n$  is empty, we suppose that there is an  $x \in r$  with  $x \in \bigcap_{n=1}^{\infty} I_n$ . Since  $0 \notin I_1$ , we can say that  $x > 0$ . But by the Lemma at the top of this set of notes, there is an  $n_1 \in \mathcal{N}$  with  $\frac{1}{n_1} < x$ , so that  $x \notin \left(0, \frac{1}{n_1}\right)$ . This contradiction proves the assertion in the example, which shows why the intervals in the theorem must be closed.

**Example 10** Can you make up a similar example using infinity? This will show that the intervals in the theorem must also be bounded.

**Example 11** This example reminds me of a “paradox” mentioned to me by a student in our class. He said it had come up in a philosophy class: Suppose that you start a pile of apples with 1 apple. Then you add 10 and take away the first. Then you add another 10 and take away the second. You continue in this fashion. I will also assume, (though he didn’t mention this) that the first step takes 1 minute, the second  $\frac{1}{2}$  minute, the third  $\frac{1}{4}$ , then  $\frac{1}{8}$ , and so forth. We will see eventually that this means that the whole process must be over after two minutes. Question: How many apples are left? Can you answer this in the same way that we proved the statement in the last example?