.Math 0450
Honors intro to analysis
Spring, 2009
Notes #6

1 Section 2.2, Absolute value

It is important to understand inequalities involving absolute value. In class we considered the inequality

$$|x-1| < |x|,$$

and discussed a graphical method of solution. A second method is to square both sides. For any $a \in R$, $|a|^2 = a^2$,by Theorem 2.2.2(a) .Hence,

$|x-1| < |x| \Leftrightarrow (x-1)^2 < x^2$ by Example 2.1.13(a). Note that this uses the fact that $|x-1| \geq 0$.

Therefore,

$$\begin{aligned} |x-1| < |x| &\Leftrightarrow x^2 - 2x + 1 < x^2 \text{ (by expanding } (x-1)^2 \text{)} \\ &\Leftrightarrow -2x + 1 < 0 \\ &\Leftrightarrow 1 < 2x \Leftrightarrow \frac{1}{2} < x. \end{aligned}$$

Note that in in some steps of this proof, I used results from section 2.1 without explanation. We will assume these results from now on, except for homework from 2.1 . However, it would be easy to make a mistake on the first step, and try to square an inequality without knowing that both terms are positive. Hence, I cited a reason there. I had to adjust a little to get everything into coherent sentences.

This is the same answer as we got graphically. If a problem says :"Find all x such that $|x-1| < |x|$." , then you can find your answer by any method you want, but for full credit you then have to prove that your answer is correct using algebra.

Here is a second example: (Corollary 2.2.4(a)) Prove that

$$||a| - |b|| \leq |a-b|.$$

First method: "cases". One method for handling absolute value inequalities is to consider various cases. For this problem, these are:

$$\begin{aligned} &a > 0, b > 0 \\ &a < 0, b > 0 \\ &a > 0, b < 0 \\ &a < 0, b < 0. \end{aligned}$$

(i) If $a \geq 0$ and $b \geq 0$, then the inequality becomes

$$|a - b| \leq |a - b| ,$$

which is certainly true.

(ii) If $a > 0$ and $b < 0$, then

$$||a| - |b|| = |-a - b| = |a + b| \leq |a| + |b|$$

while $a - b < 0$, so

$$|a - b| = b - a = |b| + |a| \quad \text{because } a < 0.$$

Combining the last two lines gives $||a| - |b|| \leq |a - b|$.

(iii) If $a > 0, b < 0$, then use (ii), substituting b for a and a for b.

(iv) If $a < 0, b < 0$, then $||a| - |b|| = ||-a| - |-b||$ and use (i) .

1.0.1 2nd method – square both sides

Method 1 was a fair amount of work. Instead, we can square the inequality, as we did earlier. We get

$$\begin{aligned} |\ |a| - |b|\ |^2 = |a|^2 - 2\,|a|\,|b| + |b|^2 = a^2 + b^2 - 2\,|ab| \\ \leq a^2 + b^2 - 2ab = (a - b)^2 . \end{aligned}$$

Here we have use the principle that

$$-|x| \leq x$$

for any $x \in R$, with $x = -ab$. Obviously this is a lot quicker. It is good to look for ways to handle several cases at once, but sometimes, considering each case separately is unavoidable or at least more straightforward.

The text uses still another method to prove this result.

The idea of an ε-neighborhood is very important. Be sure you understand the entire second half of page 33. We will use this often.

2 Completeness property

2.0.2 Read first: 2.2.3, 2.2.4.

Recall that earlier we showed that there is no rational number $r = \frac{m}{n}$ such that $r^2 = 2$. We can ask whether there is any real number r such that $r^2 = 2$. In other words, "Does $\sqrt{2}$ exist?" Can we deduce the existence of $\sqrt{2}$ from the axioms we gave earlier (A1-A4,M1-M4,D, plus order axioms (i), (ii), (iii) ?

The answer is no. This is obvious because the set Q of rational numbers obeys all of those axioms, and yet within Q, $\sqrt{2}$ does not exist. So, we need another axiom for R. This is called the "completeness" property.

Several definitions are needed to state this property Be sure you read them on page 35. We need first the terms "upper bound", "bounded above", "supremum". The latter is abbreviated "sup". Here are some examples:

Example 1 *Let* $A = \left\{\frac{m}{n} \ : \ m \in N, n \in N, m < n\right\}$. *Then* 2 *is an upper bound for* A, *so* A *is bounded above. However* 2 *is not the supremum of* A. *Instead,* $\sup A = 1$. *In this case,* $\sup A \notin A$.

Example 2 $A = \{x \in R \ : \ 0 < x \leq 2.5\}$. *Then* 147 *is an upper bound for* A . *However (obviously!),* 147 *is not the supremum of* A . *Instead,* $\sup A = 2.5$. *In this case,* $\sup A \in A$.

Example 3 $A = \{x \in R \ : \ 0 < x < 3\}$. *In this case,* $\sup A = 3 \notin A$.

A question you probably would not think to ask is: Is N bounded? Surprisingly, we can't answer that question based on the axioms we have had so far. We will need the completeness axiom, which I will now state:

Completeness axiom for R : Every subset of R which is bounded above has a supremum in R.

Notice that Q does not satisfy that property. Let $A = \{r \in Q \ : \ r^2 < 2\}$. Then $\frac{13}{2}$ is an upper bound for A (which is in Q), but A does not have a supremum in Q since, as we will see, a supremum would have to satisfy $r^2 = 2$.

Theorem 4 *$\mathcal{N}$ is an unbounded subset in R.*

Proof. Suppose that $\mathcal{N}$ is bounded. Then by the completeness axiom, $\mathcal{N}$ has a supremum, say u. Then by definition of supremum, $u - 1$ is not an upper bound for $\mathcal{N}$. There must be an integer $n \in \mathcal{N}$ such that $n > u - 1$. By 2.1.7(b), $n + 1 > u$. Hence, u is not an upper bound for $\mathcal{N}$, a contradiction. ■

This theorem is often called the "Archimedean property" : If $x \in R$, then there is an $n_x \in \mathcal{N}$ such that $n_x > x$.

Corollary (2.4.4) If $S = \left\{x \ : \ x = \frac{1}{n} \text{ for some } n \in \mathcal{N}\right\}$, then $\inf S = 0$. (see the text for the definition of inf .)

Proof: $0 \le \frac{1}{n}$ for each $n \in \mathcal{N}$, so 0 is a lower bound for S. By the completeness property, S has an infimum. Say the infimum is p. Then $p \ge 0$ because 0 is a lower bound. Suppose that $p > 0$. Then $\frac{1}{p} \in R$, so there is an $n \in N$ with $n > \frac{1}{p}$. Then $\frac{1}{n} < p$ and p is not a lower bound. Hence, $p = 0$.

In the text, these properties are used to prove that $\sqrt{2}$ exists. The proof is surprisingly complicated, and I may not have time to go over it in class.

Homework, due Thursday, January 29.

Remember to use complete sentences. Don't give a string of equations or inequalities that are not part of a logical sentence, using correct grammar.

pg. 30, # 16 (a,d), (Answers in back. Prove these answers, giving a reason for each step (each new = and each new inequality.

pg. 34, # 13(d) , 15.(give a specific ε, in terms of a and b. Prove your ε works.)

pg. 38, # 4 (brief reason, not a proof)

pg. 38, # 5. (Cite a specific reason for any step that requires a result from section 2.3. You can omit the citation when the reason is something from before section 2.3.) .

pg. 43, # 2 (answer in back; explain it) , 3, 8(a) .9(a) (answer in back; explain it.)