.Math 0450
Honors intro to analysis
Spring, 2009
Notes #5

1 Order properties of R

1.0.1 Read first: section 2.1

In addition to properties A1-A4, M1-M4, and D, we need three more axioms, called "order axioms", because they allow us to "order" the real numbers. This means, we can say that if $a \neq b$, then $a < b$ or $a > b$. These are defined, as usual, in terms of sets.

Order axioms: There is a nonempty subset $P \subset R$ such that
(i) If a and b are in P, then $a + b$ is in P.
(ii) If a and b are in P, then $a \cdot b$ is in P
(iii) For each $a \in R$, exactly one of the following is true: $a \in P$, $-a \in P$, $a = 0$.

The last property is called the "trichotomy" property of R.

Notation: If $b - a \in P$, then we write $a < b$ or $b > a$.

Remark 1 *The statement*

$$a < b < c$$

is shorthand for: $a < b$ *and* $b < c$*. In other words,* $b - a \in P$ *and* $c - b \in P$*. By property* $O1$ *(order (i)),* $(b - a) + (c - b) \in P$*, and using A2 and A1 we get* $c - a \in P$*, or* $a < c$*.* ***Never*** *write: "*$a < b > c$*", because then you have no idea how to compare* a *and* c*.*

In your homework from this section, proofs involving the order axioms may also involve the algebraic axioms. You should cite every axiom or previous result needed, whether it is of algebraic or order type. (The text doesn't bother with justifying algebraic steps in this section, but I want you to do this on the homework from this section.

Example: Solve the inequality $2x + 3 \leq 6$.

Let A be the set of all real numbers x such that $2x + 3 \leq 6$. For any $x \in R$,

$$\begin{aligned} x &= 1 \cdot x \text{ by M3} \\ &= \left(\frac{1}{2}2\right) x \text{ by M4} \\ &= \frac{1}{2}(2x) \quad \text{by M2} \\ &= \frac{1}{2}(2x + 0) \text{ by A3} \\ &= \frac{1}{2}(2x + (3 - 3)) \text{ by A4} \\ &= \frac{1}{2}((2x + 3) - 3) \text{ by A2.} \end{aligned}$$

If $2x - 3 = 6$, then

$$x = \frac{1}{2}(6 - 3) = \frac{3}{2} \text{ by arithmetic.}$$

If $2x - 3 < 6$, then

$$(2x + 3) - 3 < 6 - 3 = 3 \text{ by Theorem 2.1.7(b),}$$

so

$$x = \frac{1}{2}((2x - 3) - 3) < \frac{1}{2} \cdot 3 = \frac{3}{2} \text{ by Theorem 2.1.7(c).}$$

Hence,

$$A \subseteq \left\{x \ : x \leq \frac{3}{2}\right\}.$$

Conversely, if $x = \frac{3}{2}$, then arithmetic shows that $2x - 3 = 6$, while if $x < \frac{3}{2}$, then

$$\begin{aligned} 2x &< 3 \text{ by Theorem 2.1.7(c)} \\ 2x + 3 &< 6 \text{ by 2.1.7(b).} \end{aligned}$$

Hence,

$$A = \left\{x : \ x \leq \frac{3}{2}\right\}.$$

Notice that there was no statement about any particular number being in the set P. All we know is that $0 \notin P$. In fact, using the axioms the text proves that

$N \subseteq P$. See Theorem 2.1.8. The proof starts by proving that for any $a \in R$ which is not zero, $a^2 \in P$.

Previously we discussed the "Well-ordering" property of $\mathcal{N}$. This axiom states that "every subset of $\mathcal{N}$ has a smallest element.". Now that we have the idea of "order", we can make more sense of this. Notice that this statement does not hold for R. One of the first results in the book that is like the kind of analysis we will be doing to justify calculus is the following:

Theorem 2.1.9: If $a \in R$, and for every $\varepsilon \in P$, $0 \leq a < \varepsilon$, then $a = 0$.

In mathematical analysis, the letter ε is often used for a number which is supposed to be 'small". But there is no obvious definition of what "small" means. It is perhaps the main task in beginning analysis to understand how this idea is made rigorous.

In the statement of the theorem, there is no restriction on ε. It could be 1, or 10, or 10^{100}. But to say that "$a < \varepsilon$" is clearly more restrictive if ε is small. If $a < .0001$, then $a < 1$ automatically. The hypothesis of the theorem implies that $a < .0001$, and $a < .000001$, and $a < 10^{-1000000}$. But it also says that $a \geq 0$. The conclusion that is that a cannot be less than every number in P, and yet $a \geq 0$ unless $a = 0$. In other words, P does not have a smallest element.

The proof is fairly easy: Suppose that a is the smallest element in P. Then consider $\frac{1}{2}a$. Using the various theorems about order, it follows that $\frac{1}{2}a > 0$, but $\frac{1}{2}a < a$, a contradiction.

Be sure to go over Bernoulli's Inequality, on page 29. I probably won't have time to prove this in class, but we will use it later.

2 Absolute value and the real line

2.0.2 Read section 2.2

There is quite a lot crammed into these few pages. One important idea is that an absolute value often can be interpreted as a distance on the real line.

- If $a \in R$, then $|a|$ is the distance from a to 0.
- If $a, b \in R$, then $|a - b|$ and $|b - a|$ are both equal to the distance from a to b .

We can use this to do Example 2.2.6 (b): Find all numbers x such that $|x-1| < |x|$.

Solution: This says that the distance from x to 1 is less than the distance from x to 0. This is obviously true if $x \geq 1$, and also if $x \leq 0$. For $0 < x < 1$, by drawing a real line and marking 0 and 1, you can see that the final answer is the set $\left\{x \ : \ x > \frac{1}{2}\right\}$. The text gives two formal proofs of this.

Another important idea is an "ε-neighborhood.". This will be used a lot later. Following on with the idea of distance, we can say that

If $a \in R$, and $\varepsilon > 0$, then an ε-neighborhood of a is the set of all points whose distance from a is less than ε. In other words, as in the text, the ε- neighborhood is the set

$$V_\varepsilon(a) = \{x \in R \ : \ |x-a| < \varepsilon\}.$$

Or,

$$V_\varepsilon(a) = \{x \in R \ : \ a - \varepsilon < x < a + \varepsilon\}.$$

Try hard to understand the last theorem in the section:

Theorem: *If* $a \in R$, and $x \in V_\varepsilon(a)$ for every $\varepsilon > 0$, then $x = a$.