.Math 0450
Honors intro to analysis
Spring, 2009
Notes #4
corrected (as of Monday evening, 1/12)
some changes on page 6, as in email.

0.1 More on infinity

0.1.1 If you haven't read 1.3, do so now!

In notes#1 we gave a definition of infinity (Definition 6) which was different from that in the text. However, we showed that the two definitions are equivalent. One advantage of Definition 6 is that it doesn't rely on use of $\mathcal{N}$, and therefore doesn't depend on what axioms are used for $\mathcal{N}$.

There are several theorems given in the text about finite and infinite sets. Some of these are proved in the Appendix, and we will not cover these proofs. We can prove some of those theorems using Definition 6, and therefore not relying on use of $\mathcal{N}$.

In the proof of the following theorem we will follow the book's notation, in which $B \subset A$ means that $B \subseteq A$ and $B \neq A$. In this case we say that B is a "proper" subset of A. This use of $\subset$ is not universal.

Theorem 1 *If A is infinite, and $A \subseteq C$, then C is infinite.*

Proof. By Definition 6, notes #1, since A is infinite, there is a bijection $f : A \to B$ where $B \subset A$. If $A = C$, then there is nothing to prove. If $A \subset C$, then we define a bijection $\hat{f} : C \to D$, where $D \subset C$, as follows:

$$\hat{f}(c) = \left(\begin{array}{c} f(c) \text{ if } c \in A \\ c \text{ if } c \in C \backslash A \end{array} \right.$$

It is easy to show that this is a bijection because f is a bijection. Then

$$D = \hat{f}(C) = (C \backslash A) \cup B,$$

and since B is a proper subset of A, D is a proper subset of C. This shows that C is infinite. ■

As for finite sets, we have

Definition 2 *A set A is finite if it is not infinite.*[1]

Then, we have a corollary to Theorem 1. (A corollary is a result that is implied by a theorem and requires little or no additional proof.)

Corollary 3 *If A is finite, and $B \subseteq A$, then B is finite.*
Proof. *Use proof by contradiction. If B is infinite, then Theorem 1 implies that A is infinite, a contradiction of our hypothesis in the Corollary.* ■

The corollary can be called the "contrapositive" of the theorem. But I will not place much emphasis on terms like this. They are discussed in Appendix A.

The most important infinite set is $\mathcal{N}$. There are many bijections from $\mathcal{N}$ to proper subsets of itself, in accordance with Definition 6. Among the finite sets, the text frequently makes use of the sets

$$\mathcal{N}_n = \{1, 2, 3, ..., n\}$$

where $n \in \mathcal{N}$.

[1] Compare this with the text, where finite is defined first, and then a set is said to be infinite if it is not finite. Since we showed the two definitions of infinite are equivalent, the two definitions of finite must be as well.

1 Countable sets

One of the major discoveries of 19th century mathematics was Cantor's work on infinity. After Cantor, mathematicians felt that they really understood this concept. And we will too, after the next few pages!

They key is the idea of a "cardinality", and a related concept, "cardinal number". A cardinal number can be thought of as an extension of $\mathcal{N}$. In particular, each element of $\mathcal{N}$ is a different cardinal number. And the cardinality of a set is the cardinal number of the elements in the set. For example, the cardinality of the set $\{1, 4, 6, 7, 9\}$ is 5, as is also the cardinality of the set $\{a, b, c, d, e\}$. We have the following important definition:

Definition 4 *Two sets A and B are said to have the same "cardinality" if there is a bijection $f : A \rightarrow B$.*

Thus, the function $\{(a, 1), (b, 4), (c, 6), (d, 7), (e, 9)\}$ is a bijection between the two sets above, and obviously, there is a bijection between each of these sets and the set $N_5 = \{1, 2, 3, 4, 5\}$.

Clearly, the set $\mathcal{N}$ has a different cardinality from N_5, or $\mathcal{N}_{1,000,000}$, or N_n for any n. There is a special word applied to sets with the same cardinality as $\mathcal{N}$.

Definition 5 *Any set with the same cardinality as $\mathcal{N}$ is called "denumerable". Any set which is either finite or denumerable is called "countable". A denumerable set may also be called "countably infinite".*

Some authors use "countable" to mean "countably infinite". I will try to follow the text, however, which uses the definition above.

Here are some obvious countably infinite sets.

1. The set of even integers. Clearly, $f(n) = 2n$ is a bijection between the even numbers and $\mathcal{N}$. But notice an important implication: We can have sets A and B, with $A \subset B$, and yet A and B have the same cardinality. (Recall that $\subset$ implies that the sets are not equal. A is a proper subset of B.) No pair of finite sets can have this property, by the definitions we gave of infinite and finite sets.

2. The set of odd integers. Here, $f(n) = 2n - 1$.

3. The set
$$\left\{ x \in R \ : \ x = \frac{1}{n} \text{ for some } n \in N \right\}.$$

Here is a less obvious example:

Theorem 6 *(Theorem 1.3.8 in the text) : The Cartesian product* $\mathcal{N} \times \mathcal{N}$ *is countable.*

This is much less obvious than the previous examples. Somehow we think that $\mathcal{N} \times \mathcal{N}$ is a "larger" set than $\mathcal{N}$. And in one sense, it is, since the set

$$A = \{(n, 1) \in \mathcal{N} \times \mathcal{N}\}$$

obviously has the same cardinality, and $A \subset \mathcal{N} \times \mathcal{N}$. But we saw in example **1** that this is no reason that A and $\mathcal{N} \times \mathcal{N}$ can't have the same cardinality.

Proof. The text gives what it calls an "informal" proof on page. 18, and relegates the formal proof to the appendix. This means that the specific bijection between A and $\mathcal{N} \times \mathcal{N}$ is given by a rather complicated formula back in the appendix. In the chapter, it is merely indicated by a diagram. I will give a different diagram, and consider this an adequate proof, since I think everyone will be convinced. Here is a diagram that shows the bijection, and also the first few ordered pairs in the bijection.

$$\begin{array}{ccccccc} & & & (1,1) & & & \\ & & (1,2) & & (2,1) & & \\ & (1,3) & & (2,2) & & (3,1) & \\ (1,4) & & (2,3) & & (3,2) & & (4,1) \end{array}$$

Notice that in the first row are all the ordered pairs $(m, n) \in \mathcal{N} \times \mathcal{N}$ with $m+n = 2$. In the second row are all the ordered pairs $(m, n) \in \mathcal{N} \times \mathcal{N}$ with $m + n = 3$. In the third row, $m + n = 4$, and so forth. In this way, all possible ordered pairs are included eventually. And our bijection (say $f : \mathcal{N} \to \mathcal{N} \times \mathcal{N}$) is given by listing the rows of the above "Pascal's triangle" from the top down, and left to right within a row.

$$\{(1, (1, 1)), (2, (1, 2)), (3, (2, 1)), (4, (1, 3)), (5, (2, 2)), (6, (3, 1)), ...\}$$

Notice that the ordered pairs in this function are more complicated than before. The domain is $\mathcal{N}$, so the first element in the ordered pair is a positive integer. The image set is $\mathcal{N} \times \mathcal{N}$, so the second element of the ordered pair is itself an ordered pair. ■

You might find it an interesting exercise to come up with a formula for $f(n)$. Maybe you can find something simpler than what is given in appendix, on page 344. There they do the bijection in the other way, from $\mathcal{N} \times \mathcal{N} \to \mathcal{N}$. Can you prove that if $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is also a bijection? We will use this result several times.

Corollary 7 *The set Q^+ of positive rational numbers $\frac{m}{n}$ is countable.*
Proof. *We use almost the same bijection as in the proof above of Theorem 1.3.8. But in considering the rows of Pascal's triangle, we remove all pairs (m, n) where the fraction $\frac{m}{n}$ is not in "lowest terms". This means all pairs (m, n) such that m and n have a common integer factor greater than 1. Each row has at least one element which is in lowest terms, namely the first, $\frac{1}{n}$. Hence we get an infinite set of elements in the remaining parts of the triangle, and we then "count" them as before, to give the desired bijection from $\mathcal{N}$ to Q^+.* ■

Remark 8 *The text does the last two theorems somewhat differently. Either approach is ok; use whichever you understand better.*

From this it is an easy step to showing that the set Q of all rational numbers is countable. This follows from a much more general theorem:

Theorem 9 *(Theorem 1.3.12.) If for each $m \in \mathcal{N}$, A_m is a countable set, then $\cup_{m\in\mathcal{N}} A_m$ is a countable set.*

Proof. (Again, the text gives a different proof. Then, in a remark at the bottom of page 29, they give essentially the proof I will describe.) For each m, there is a bijection f_m, either from $\mathcal{N}$ to $\mathcal{A}_m$, or (if A_m is finite), from $\mathcal{N}_n$ to A_m for some n. Consider again a Pascal's triangle, arranged as follows:

$$
\begin{array}{ccccccc}
 & & & f_1(1) & & & \\
 & & f_1(2) & & f_2(1) & & \\
 & f_1(3) & & f_2(2) & & f_3(1) & \\
f_1(4) & & f_2(3) & & f_3(2) & & f_4(1)
\end{array}
$$

and so forth. If some of these aren't defined, because some A_m is a finite set, simply leave their space blank. Then we get a bijection from $\mathcal{N}$ to $\cup_{m\in\mathcal{N}} A_m$ as follow:

$$\{(1, f_1(1)), (2, f_1(2)), (3, f_2(1)), (4, f_1(3)), ...\}$$

moving down the rows, and when reaching a row, moving from left to right across the row. ■

To show that Q is countable, we have $Q = Q^+ \cup Q^- \cup \{0\}$. This is only a finite union, which means all elements in the triangle involving f_4 or higher are omitted.

So far every set we have seen has been either finite or denumerable. So every set has been countable. We can ask whether every infinite set is denumerable. One of Cantor's most important theorems helps us answer that question. For this theorem we need a definition.

Definition 10 *If A is a set, then the "power set" of A is the set $P(A)$ of subsets of A.*

Example 11 *Suppose that $A = \{1, 2, 3\}$. Then*

$$P(A) = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, A, \phi\}$$

It is important to note that both A and ϕ are elements of the set $P(A)$.

Theorem 12 *If A is a finite set with n elements, then $P(A)$ has 2^n elements.*
Proof. *We will use induction. It is sufficient to assume that $A = N_n$, for if not, then there is a bijection from A to N_n. The theorem is true for $n = 1$, since the power set of $\{1\}$ is $\{\{1\}, \phi\}$. Suppose that it is true when $n = k$.*

Then consider N_{k+1}. We have

$$N_{k+1} = N_k \cup \{k + 1\}.$$

Since the theorem is true for $n = k$, N_k has 2^k subsets, each of which is also a subset of N_{k+1}. For each subset S of N_k, there is an additional subset $S \cup \{k + 1\}$ of N_{k+1}. Thus, the total number of subsets of N_{k+1} is $2\left(2^k\right) = 2^{k+1}$. Thus, the theorem is true for all n. ■

This is pretty simple. One thing it shows is that the power set of a finite set has a different cardinality from the original set, and also from any subset of the original set Now here is Cantor's theorem

Theorem 13 *If A is any set, then there is no surjection of A onto $P(A)$.*

Corollary 14 *If A is any set, then $P(A)$ has a different cardinality from A.*

Proof. Suppose that $f : A \to P(A)$ is a surjection. Consider some $a \in A$. $f(a)$ is some subset of A. Either $a \in f(a)$ or $a \notin f(a)$. Let

$$D = \{a \in A \mid a \notin f(a)\}.$$

Certainly D is a subset of A. Since f is a surjection, there is some $a_0 \in A$ such that $f(a_0) = D$. Then either, $a_0 \in D$ or $a_0 \notin D$. If $a_0 \in D$, then $a_0 \in f(a_0)$, which means (from the definition of D) that $a_0 \notin D$, a contradiction. If $a_0 \notin D$, then $a_0 \in f(a_0) = D$, again a contradiction. This means that no such surjection can exist. ■

The implication of this is interesting: There is a set which is "bigger" than $\mathcal{N}$, meaning that there is no bijection from this set to $\mathcal{N}$ or to any subset of $\mathcal{N}$. Thus, there are "uncountable sets". We will see in the next chapter that $\mathcal{R}$ is uncountable.

Homework, due at the beginning of class on January 21

Remark: I find that I must change my homework policy a bit. While I will continue to give 5-10 problems, I cannot promise that any beyond 5 will be short and easy. Otherwise we won't cover the important material in the homework.

pg. 15, # 3, 7,20
pg. 21, # 4, 12
pg. 29, # 1(a), 3(c), 7, 10(b)