.Math 0450
Honors intro to analysis
Spring, 2009
Notes #3

0.1 More on infinity

In notes#1 we gave a definition of infinity (Definition 6) which was different from that in the text. However, we showed that the two definitions are equivalent. One advantage of Definition 6 is that it doesn't rely on use of $\mathcal{N}$, and therefore doesn't depend on what axioms are used for $\mathcal{N}$.

There are several theorems given in the text about finite and infinite sets. Some of these are proved in the Appendix, and we will not cover these proofs. We can prove some of those theorems using Definition 6, and therefore not relying on use of $\mathcal{N}$.

In the proof of the following theorem we will follow the book's notation, in which $B \subset A$ means that $B \subseteq A$ and $B \neq A$. In this case we say that B is a "proper" subset of A. This use of $\subset$ is not universal.

Theorem 1 *If A is infinite, and $A \subseteq C$, then C is infinite.*

Proof. By Definition 6, notes #1, there is a bijection $f : A \to B$ where $B \subset A$. If $A = C$, then there is nothing to prove. If $A \subset C$, then we define a bijection $\hat{f} : C \to D$, where $D \subset C$, as follows:

$$\hat{f}(c) = \left(\begin{array}{c} f(c) \text{ if } c \in A \\ c \text{ if } c \in C \backslash A \end{array} \right.$$

It is easy to show that this is a bijection because f is a bijection. Then

$$D = \hat{f}(C) = (C \backslash A) \cup B,$$

and since B is a proper subset of A, D is a proper subset of C. This shows that C is infinite. ∎

As for finite sets, we have

Definition 2 *A set A is finite if it is not infinite.*[1]

[1] Compare this with the text, where finite is defined first, and then a set is said to be infinite if it is not finite. Since we showed the two definitions of infinite are equivalent, the two definitions of finite must be as well.

Then, we have a corollary to Theorem 1. (A corollary is a result that is implied by a theorem and requires little or no additional proof.)

Corollary 3 *If A is finite, and $B \subseteq A$, then B is finite.*
Proof. *Use proof by contradiction. If B is infinite, then Theorem 1 implies that A is infinite, a contradiction of our hypothesis in the Corollary.* ■

Remark 4 *The corollary can be called the "contrapositive" of the theorem. But I will not place much emphasis on terms like this. They are discussed in Appendix A.*

The most important infinite set is $\mathcal{N}$. There are many bijections from $\mathcal{N}$ to proper subsets of itself, in accordance with Definition 6. Among the finite sets, the text frequently makes use of the sets

$$\mathcal{N}_n = \{1, 2, 3, ..., n\}$$

where $n \in \mathcal{N}$.