Math 0450
Honors intro to analysis
Spring, 2009
Notes 21

1 Exponentials and logs

read first: 8.3. Recall that in Chapter 3 (page 73) we had the definition:

$$e = \lim \left(1 + \frac{1}{n}\right)^n .$$

We saw that $2 < e < 3$. Referring to pg. 74, we see that the binomial theorem is used to get

$$\left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\frac{1}{4!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)+\cdots$$

Since the products $\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)$, etc. are all less than 1, we see that

$$\left(1+\frac{1}{n}\right)^n < 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$$

Hence,

$$e \leq \sum_{j=0}^{\infty} \frac{1}{j!}$$

On the other hand, look at the coefficient of $\frac{1}{3!}$, for example. It is $\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)$. This increases as n increases (since less is being subtracted from 1 in each factor) and

$$\lim \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) = 1.$$

So we can see, for example, that

$$e = \lim \left(1+\frac{1}{n}\right)^n \geq \lim \left(1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\right) = 1+1+\frac{1}{2!}+\frac{1}{3!}$$

Similarly, for any m,

$$e \geq \sum_{j=0}^{m} \frac{1}{j!}$$

Hence,

$$e = \sum_{j=0}^{\infty} \frac{1}{j!}.$$

In section 8.3 e is defined in a different way, after defining the function $f(x) = e^x$. From the previous definition of e, we know what e^2, e^3, $e^{\frac{2}{3}}$, etc. are, but we don't, for example, know what e^π might be. In 8.3, e is defined without referring to the previous definition. Fortunately, the two definitions turn out to be the same.

The function e^x is defined using differential equations. We look for a function $E(x)$ which is differentiable and satisfies the following two conditions:

$$E'(x) = E(x) \text{ for all values of } x \tag{1}$$
$$E(0) = 1. \tag{2}$$

We can rightly ask if any such function exists. It is proved that such a function does exist by developing a formula for $E(x)$ in terms of an infinite series.

This is done by integrating $(1) - (2)$ to give

$$E(x) - E(1) = \int_0^x E \tag{3}$$

It turns out that $E(x)$ is the limit of the following sequence of functions:

$$\begin{aligned}
f_0(x) &= 1 \\
f_1(x) &= 1 + x \\
f_2(x) &= 1 + x + \frac{1}{2!}x^2 \\
f_3(x) &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \\
&\cdots \\
f_n(x) &= \sum_{i=0}^{n} \frac{1}{i!}x^i
\end{aligned}$$

To see this, we first have to prove that this sequence of functions converges. The text does this, showing convergence on $(-\infty, \infty)$ and uniform convergence on $[-K, K]$ for any K.

To verify that the resulting function E satisfies the integral equation (3), we start by differentiating the equations for $f_n(x)$. For example,

$$\begin{aligned} f_3'(x) &= 1 + x + \frac{1}{2!}x^2 = f_2(x) \\ f_n' &= f_{n-1}, \end{aligned}$$

and so

$$\int_0^x f_{n-1} = \int_0^x f_n' = f_n(x) - f_n(0) = f_n(x) - 1$$

Then we have

$$E(x) = \lim f_n(x) = 1 + \lim \int_0^x f_{n-1}.$$

Since (as is shown in the text), the sequence f_n (and so, the sequence f_{n-1}) converges uniformly, we can interchange the limit and the integral, to get

$$E(x) = 1 + \int_0^x \lim f_{n-1} = 1 + \int_0^x E.$$

Thus, E satisfies (3). Setting $x = 0$ gives (2), and differentiating both sides of (3), we get (1).

We also have to show that there is only one solution. (Otherwise, there could be two different values of e.) We can prove this using Taylor's theorem with remainder, discussed in notes 19.

Theorem 1 *The solution $E(t)$ of $(1) - (2)$ is unique.*

Proof. Suppose that there are two solutions, say E_1 and E_2. Let $H = E_1 - E_2$. Then

$$\begin{aligned} H' &= E_1' - E_2' = E_1 - E_2 = H \\ H(0) &= E_1(0) - E_2(0) = 0. \end{aligned}$$

For some $X > 0$, H is continuous on $[0, X]$, and H' exists on this interval. Furthermore,

$$\begin{aligned} H'' &= H' = H \\ H''' &= H'' = H' = H, \end{aligned}$$

and so forth, so every derivative of H exists. In particular, for each j, the j^{th} derivative $H^{(j)}$ of H satisfies

$$H^{(j)}(0) = H(0) = 0.$$

Applying Taylor's theorem with remainder, we get, for any $n \in \mathcal{N}$, that

$$\begin{aligned} H(X) &= \frac{1}{n!}\int_0^X H^{(n+1)}(t)(X-t)^n\,dt = \frac{1}{n!}\int_0^X H(t)(X-t)^n\,dt \le \frac{\max_{0\le t\le X}|H(t)|}{n!}\frac{X^{n+1}}{n+1} \\ &= \max_{0\le t\le X}|H(t)|\frac{X^{n+1}}{(n+1)!}. \end{aligned}$$

Letting $n \to \infty$, with X fixed, we can use exercise 3.2.18(c) in the text to say that $H(X) = 0$. This is true for any X, so $H = 0$ and $E_1 = E_2$.

■

The number e is then defined to be $E(1)$. It is obvious from our calculation above that this is the same as before:

$$e = E(1) = \sum_{i=0}^{\infty}\frac{1}{n!}1^n = \sum_{i=0}^{\infty}\frac{1}{n!} = \lim\left(\left(1+\frac{1}{n}\right)^n\right).$$

The rules satisfied by exponents are also proved in the text. Here is one:

Theorem 2 *For any x and y, $E(x+y) = E(x)E(y)$.*

Proof. Suppose $y \in R$. (We hold y "fixed.") Let

$$f(x) = \frac{E(x+y)}{E(y)}.$$

Then

$$\begin{aligned} f'(x) &= \frac{E'(x+y)}{E(y)} = \frac{E(x+y)}{E(y)} = f(x) \\ f(0) &= \frac{E(y)}{E} = 1. \end{aligned}$$

Hence, by the earlier uniqueness theorem, $f(x) = E(x)$, and from the the definition of $f(x)$, this proves the theorem. ■

1.0.1 Definition of natural logarithm

Since $E' = E$ and $E(0) = 1$, it is seen that E is strictly increasing, at least for $x \geq 0$. Since $E'(-x) = E(-x) = \frac{1}{E(x)}$, E is an increasing function on $(-\infty, \infty)$. Hence, it has an inverse function, which we will call L. As an inverse, it satisfies

$$\begin{aligned} &L(E(x)) = L(e^x) = x \text{ for any } x \in R \\ &E(L(x)) = x \text{ for any } x > 0. \end{aligned}$$

and from this we see that L is the natural logarithm:

$$L(x) = \ln x.$$

The usual rules for logarithm follow, such as $L(xy) = L(x) + L(y)$. This is because $E(L(x) + L(y)) = E(L(x))\,E(L(y)) = xy$.

From this, for example we get

$$L\left(x^2\right) = 2 \ln x.$$

1.1 Definition of a^x when $a > 0$.

Now we have defined the value of e^x for every real number x. We can use this to define other exponents, such as 2^x. We set

$$2^x = e^{x \ln 2}.$$

To see that this makes sense, we have to show it is correct when x is an integer, or rational. For example, from this definition,

$$2^2 = e^{2 \ln 2} = e^{\ln 4} = 4.$$

Remark 3 *It is interesting that such a number as 2^π is defined using calculus. It could be defined in other ways, for example as*

$$\lim\left(2^3, 2^{31/10}, 2^{314/100} \ldots\right)$$

But any definition requires a limit process, which is at the heart of calculus.

1.2 Derivative of $\ln x$.

By the Theorem 6.1.8, which is the formula for the derivative of an inverse function, we have

$$L'(E(x)) = \frac{1}{E'(x)} = \frac{1}{E(x)}.$$

If $E(x) = y$, then $L'(y) = \frac{1}{y}$.

1.3 Taylor series for $\ln x$

We can't find a Taylor series around $x = 0$ for L, because $L(0)$ is not defined. Instead, we can let $f(x) = L(1+x)$. We find that

$$L(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots \tag{4}$$

2 Trig functions

The same technique is used to define sin and cos, but now the differential equation is "second order", because it involves a second derivative. $\sin x$ is defined to be the solution of

$$\begin{aligned} S'' + S &= 0 \\ S(0) = 0, S'(0) &= 1 \end{aligned}$$

and $\cos x$ is the solution of

$$\begin{aligned} C'' + C &= 0 \\ C(0) = 1, C'(0) &= 0. \end{aligned}$$

I think it is quite amazing that all of the properties of sine and cosine follow. We start with the Taylor series, by finding derivatives at 0:

$$\begin{aligned} S(0) = 0, S'(0) &= 1 \\ S''(0) = -S(0) &= 0 \\ S'''(0) = -S'(0) &= -1 \\ &\cdots \end{aligned}$$

so for any n,

$$S(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots + \frac{1}{(2n+1)!}\int_0^x S^{2n+1}(t)(x-t)^{2n}\,dt$$

We showed in notes 19 that the remainder term can also be written as $(-1)^n \frac{S^{2n+1}(c)}{(2n+1)!}x^{2n+1}$ for some c between 0 and x.

By differentiating the Taylor series, we see that

$$S' = C, C' = -S.$$

Now we can prove:

Theorem 4 $S(x+y) = S(x)C(y) + C(x)S(y)$.

Proof. Suppose that $y \in R$, and let

$$f(x) = S(x+y), g(x) = S(x)C(y) + C(x)S(y).$$

Then:

$$\begin{aligned} f'(x) &= S'(x+y) \\ f''(x) &= S''(x+y) = -S(x+y) = -f \end{aligned}$$

and

$$\begin{aligned} g'(x) &= S'(x)C(y) + C'(x)S(y) = C(x)C(y) - S(x)S(y) \\ g''(x) &= C'(x)C(y) - S'(x)S(y) = -S(x)C(y) - C(x)S(y) = -g(x) \end{aligned}$$

Finally,

$$f(0) = S(y), f'(0) = C(y)$$
$$g(0) = S(y), g'(0) = C(y)$$

and so by a uniqueness theorem similar to the one we proved for E, $f = g$.

■

We can also prove that

$$S^2 + C^2 = 1 : \tag{5}$$

$$\begin{aligned} \left(S^2 + C^2\right)' &= 2SS' + 2CC' = 2SC + 2C(-S) = 0 \\ S(0)^2 + C(0)^2 &= 1. \end{aligned}$$

We can use these functions to define π, and show that C and S have period 2π. See pages 250-251 of the text.