Math 0450
Honors intro to analysis
Spring, 2009
Notes 20

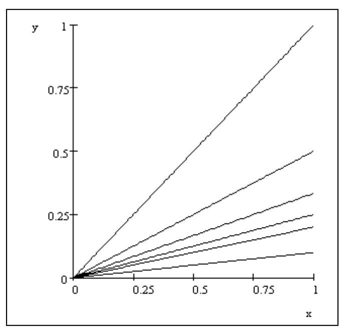
1 Uniform convergence

read first: 8.1, 8.2 except for the proof of Theorem 8.2.4, which depends on chapter 7. Suppose that $\{f_n\}$ is a sequence of real valued functions defined on the interval $[0,1]$, and that f is also such a function. We say that $\{f_n\}$ converges to f if, for each x in $[0,1]$,

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Examples:

(i) $f_n(x) = \frac{x}{n}$ for $0 \le x \le 1$. Then $\{f_n\}$ converges to the function $f(x) = 0$ on the entire interval $[0,1]$. Here is a graph indicating this:

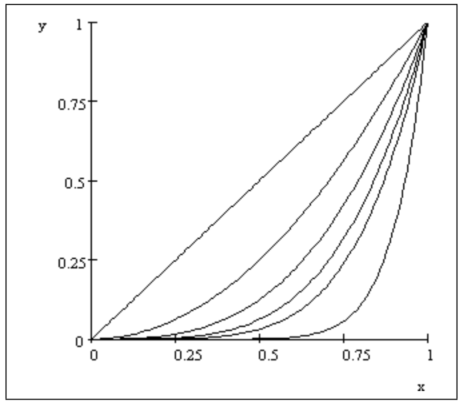


I plotted $x, \frac{1}{2}x, \frac{1}{3}x, \frac{1}{4}x, \frac{1}{5}x$ and $\frac{1}{10}x$.

(ii) $f_n(x) = x^n$ for $0 \le x \le 1$. Then $\{f_n(x)\}$ converges to the following function f :

$$f(x) = \begin{cases} 0 \text{ if } 0 \le x < 1 \\ 1 \text{ if } x = 1. \end{cases}$$

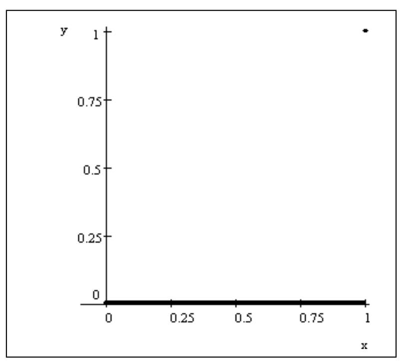
Here is the same sort of graph, again for the cases $n = 1, 2, 3, 4, 5, 10$.



What you see is that $\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$, and $\lim_{n\to\infty}\left(\frac{99}{100}\right)^n = 0$, but the first sequence converges much faster than the second. The other thing to notice is that in this case, all of the functions f_n are continuous, but the limit f is not continuous. And here is another thing to look at:

$$\int_0^1 x^n dx = \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1}.$$

Hence, $\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = 0$. Now we can ask about the integral of f. Recall the graph of f.



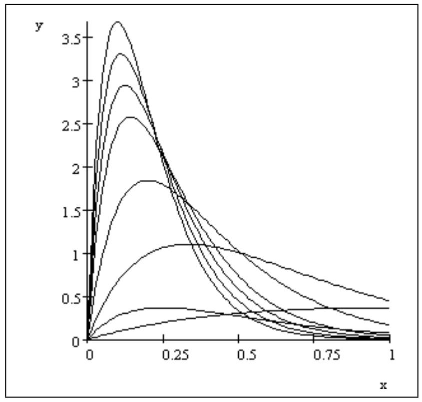
(Notice the single point at $(1,1)$.) Now we ask: What is $\int_0^1 f(x)\,dx$? For homework you had a similar problem except that the exceptional point was at $x = \frac{1}{2}$. And just as in that problem, you can prove that $\int_0^1 f(x)\,dx = 0$.

For this sequence of functions, we can either integrate first and then take the limit, or take the limit and then integrate. We get the same answer either way.

(iii) Let

$$f_n(x) = n^2 x e^{-nx} \tag{1}$$

Once again I plot several, this time for $n = 1, 2, 3, 5, 7, 8, 9, 10$.



It's not so clear what is going on. But we know that $e > 2$, so

$$n^2 x e^{-nx} = x \frac{n^2}{e^{nx}} \leq x \frac{n^2}{(2^x)^n}.$$

There are two cases. If $x = 0$ then $n^2 x e^{-nx} = 0$. If $x > 0$, then $a = \frac{1}{2^x} < 1$, and in problem 3.2.1 it is shown that $\lim (n^2 a^n) = 0$. Thus, ,

$$\lim_{n \to \infty} f_n(x) = 0 \tag{2}$$

for every $x \geq 0$.

But, what about those graphs, which seem to show the functions growing? To see what is happening, we can calculate the maximum of each of the functions f_n. We find that the maximum occurs at $x = \frac{1}{n}$. And,

$$f_n\left(\frac{1}{n}\right) = n^2 \frac{1}{n} e^{-1} = \frac{n}{e}. \tag{3}$$

So the maxima go to infinity. Can you see how (2) and (3) are both possible? If we take a **fixed** x, then $f_n(x) \to 0$ as $n \to \infty$. But if we take a different x for each n (that is, $x = \frac{1}{n}$), then we get equation (3).

The integral can be done, using integration by parts, to give

$$\int_0^1 x n^2 e^{-nx} dx = 1 - n e^{-n} - e^{-n}.$$

Since $n e^{-n} \leq n^2 a^n$ where $a = \frac{1}{e}$, $\lim_{n \to \infty} n e^{-n} = 0$. Therefore,

$$\lim_{n \to \infty} \int_0^1 f_n(x)\, dx = 1 \neq \int_0^1 \lim_{n \to \infty} f_n(x)\, dx,$$

since the limit inside the integral is 0. We wish to determine when the integration and limit operations can be done in different orders and still get the same answer.

Definition 1 *We say that a sequence of continuous functions* $\{f_n\}$ *on* $[a,b]$ *converges* ***uniformly*** *on* $[a,b]$ *to the function* f *if for every* $\varepsilon > 0$*, there is a* $K(\varepsilon)$ *such that if* $n \geq K(\varepsilon)$*, then*

$$|f_n(x) - f(x)| < \varepsilon$$

for every $x \in [a,b]$*.*

We can state this differently after the following definition:

Definition 2 *Suppose that* f *is continuous on* $[a,b]$*. Then the "norm of* f*" is*

$$||f|| = \max_{a \leq x \leq b} |f(x)|.$$

This exists because a continuous function on a closed bounded interval has a maximum and a minimum. From the definition of limit we now see that (f_n) converges to f uniformly on $[a,b]$ if and only if

$$\lim(||f_n - f||) = 0.$$

This is Lemma 8.1.8.

In example (i) above, f_n converges uniformly on $[0,1]$ to its limit function 0, but that the sequences in the other two examples do not converge uniformly. For example, in example (iii), $f_n(x) = n^2 x e^{-nx}$, and $f(x) = 0$ (since we showed that $\{f_n(x)\}$ converges to 0 for each x, but by finding the maximum of each f_n, we saw that

$$||f_n - f|| = \frac{n}{e},$$

so f_n does **not** converge uniformly to f .

The following two theorems are important applications of uniform continuity. We will use them in the next section to define the exponential function e^x (although this function can be defined in other ways as well).

Theorem 3 *If* $\{f_n\}$ *converges uniformly to* f *on* $[0,1]$ *, then* f *is continuous on* $[0,1]$*.*

This is Theorem 8.2.2, pg. 234 of B&S.

Proof. Suppose that $c \in [a,b]$. We will prove that f is continuous at c.

Suppose that $\varepsilon > 0$. Then there is a $K(\varepsilon) > 0$ such that if $n \geq K(\varepsilon)$, then $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for each x in $[a,b]$. In particular, this is true if $n = K(\varepsilon)$. Since $f_{K(\varepsilon)}$ is continuous at c, there is a δ such that if x is in $[a,b]$, and $|x-c| < \delta$, then $\left|f_{K(\varepsilon)}(x) - f_{K(\varepsilon)}(c)\right| < \frac{\varepsilon}{3}$. Hence, if $|x-c| < \delta$, then

$$\begin{aligned} |f(x) - f(c)| &= \left(f(x) - f_{K(\varepsilon)}(x)\right) + \left(f_{K(\varepsilon)}(x) - f_{K(\varepsilon)}(c)\right) + \left(f_{K(\varepsilon)}(c) - f(c)\right) \leq \left|f(x) - f_{K(\varepsilon)}(x)\right| + \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence, f is continuous at c, proving the theorem. ■

Remark 4 *From a Theorem 5.4.3, we know that f is uniformly continuous on $[a,b]$.*

Theorem 5 *If $\{f_n\}$ is a sequence of functions which are all Riemann integrable on $[a,b]$, and $\{f_n\}$ converges uniformly on $[a,b]$ to a function f , then f is Riemann integrable on $[a,b]$, and*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

This is Theorem 8.2.4. The proof in the text uses the definition of integral in chapter 7, so here is a proof using the definition in the notes on integration.

Remark 6 *There is no assumption that the f_n are continuous, and so the f need not be continuous either. Your homework included an example of a function which was not continuous but was integrable. But there are less trivial ones, such as monotone functions. Recall that any monotone function is integrable, even if it has jump discontinuities.*

Proof. The trick here is to avoid confusing the limit of f_n as $n \to \infty$ with the size of the partition used in the upper and lower sums. So for each $k \in \mathcal{N}$, let P_k denote the partition of $[a,b]$ into equal intervals of length $\frac{b-a}{k}$. Suppose that $\varepsilon > 0$. Let $\varepsilon_1 = \frac{\varepsilon}{5}$. Since (f_n) converges to f uniformly on $[a,b]$, there is a $K(\varepsilon_1)$ such that if $n \geq K(\varepsilon_1)$, then

$$|f_n(x) - f(x)| < \frac{\varepsilon_1}{b-a} \tag{4}$$

for each $x \in [a,b]$. We now fix such an n (for example, $n = K(\varepsilon)$). Since f_n is integrable, it is bounded, and (4) implies that f is bounded as well.

Also, each f_n is integrable. Let M_j and m_j be the sup and inf of f in I_j. From (5) we see that

$$f_n(x) - \frac{\varepsilon_1}{b-a} < f(x) < f_n(x) + \frac{\varepsilon_1}{b-a}. \tag{5}$$

Also let $M_{j,n}$ and $m_{j,n}$ be the sup and inf of f_n in I_j. We then have from (4) and (5), that

$$M_j \leq M_{j,n} + \frac{\varepsilon_1}{b-a}.$$

(Why?) Similarly,

$$m_j \geq m_{j,n} - \frac{\varepsilon_1}{b-a}.$$

(This is a lot simpler than what I did in class, where I introduced the point at which the functions reach their maximum. If fact, it does not even matter if the functions ever reach their supremum. We are not assuming that any of the functions are continuous, so we can't use uniform continuity either. Yet, it's simpler.)

Hence, summing the expressions for $L(P_k, f_n)$, $L(P_k, f)$, $U(P_k, f_n)$, and $U(P_k, f)$, we find that for each sufficiently large k,

$$L(P_k, f_n) - 2\varepsilon_1 < L(P_k, f) < U(P_k, f) < U(P_k, f_n) + 2\varepsilon_1. \tag{6}$$

Again, keep in mind that n is being held fixed at this stage. Since f_n is integrable, we can choose k so large that $U(P_k, f_n) - L(P_k, f_n) < \varepsilon_1$, by the Cauchy criterion for integrals. Hence,

$$U(P_k, f) - L(P_k, f) < 5\varepsilon_1 = \varepsilon.$$

This shows that f is integrable.

Recall that

$$L(P_k, f) \leq \int_a^b f \leq U(P_k, f),$$

and so from (6), for each sufficiently large k,

$$L(P_k, f_n) - 2\varepsilon_1 < \int_a^b f < U(P_k, f_n) + 2\varepsilon_1.$$

Also, $\lim_{k\to\infty} L(P_k, f_n) = \lim_{k\to\infty} U(P_k, f_n) = \int_a^b f_n$, so

$$\int_a^b f_n - 2\varepsilon_1 \leq \int_a^b f \leq \int_a^b f_n + 2\varepsilon_1$$

$$-2\varepsilon_1 \leq \int_a^b f - \int_a^b f_n \leq 2\varepsilon_1.$$

We recall that n was determined by ε_1 and so by ε. It follows that

$$\lim \left(\int_a^b f_n \right) = \int_a^b f$$

proving the theorem. ■

2 Homework, due April 16

pg. 183, # 9(a) (take logs)
pg. 233, # 10
pg. 239, # 1,14
pg. 246, # 5.