

Supplementary notes on chapter 1

January 8, 2009

Math 0450
Honors intro to analysis
Spring, 2008

(Sometimes, after class, I will decide to add to the previous notes. These updates will be dated so that you can be sure you have the latest (compare what you have with the date of posting on the webpage).

In class, I started, but did not complete, problem 22, pg. 12. Here is a complete solution.

Problem: Let f, g be functions such that $(g \circ f)(x) = x$ for all $x \in D(f)$ and $(f \circ g)(y) = y$ for all $y \in D(g)$.

Prove that $g = f^{-1}$. (And also, though the problem doesn't say this explicitly, $f = g^{-1}$.)

Solution: First we must show that f and g are injections. To show that f is an injection, suppose that x_1 and x_2 are in $D(f)$ and $f(x_1) = f(x_2)$. Then by our hypotheses, $(g \circ f)(x_1) = x_1$ and $(g \circ f)(x_2) = x_2$. But $f(x_1) = f(x_2)$, so $g(f(x_1)) = g(f(x_2))$, and so $x_1 = x_2$.

Remark: In class I used proof by contradiction. That is ok also, but a little longer and harder to understand. There are usually several ways to give a proof. Sometimes a proof by contradiction is called an "indirect" proof. It is often quicker to use a direct proof when possible.

The proof that g is injective is similar (and will be omitted).

Next we show that f and g are surjective, with $f : D(f) \rightarrow D(g)$ and $g : D(g) \rightarrow D(f)$. Suppose that $x \in D(f)$. Then $(g \circ f)(x) = x$, which only makes

sense if $f(x) \in D(g)$. Hence, $f : D(f) \rightarrow D(g)$. (I forgot this step in class.) Now we must show that f is surjective. Suppose that $y \in D(g)$. Then by our hypothesis, $(f \circ g)(y) = y$, or $f(g(y)) = y$. If we set $x = g(y)$, then $f(x) = y$. Since y is an arbitrary point in $D(g)$, this shows that f is surjective. The proof that g is surjective is similar.

Finally, we must show that $g = f^{-1}$ and $f = g^{-1}$. Suppose that $(a, b) \in f$. Then $b = f(a)$. By our hypothesis, $g(b) = (g \circ f)(a) = a$, showing that $(b, a) \in g$. Similarly, if $(b, a) \in g$, then $(a, b) \in f$. This shows that f and g are inverses of each other.