Math 0450
Honors intro to analysis
Spring, 2009
Notes 19

1 Further remarks about integrals

In class we discussed the proof from Prof. Beatrous's notes of the inequality

$$\left| \int_a^b f \right| \leq \int_a^b |f| \,.$$

We pointed out that the hard part is proving that if f is integrable on $[a,b]$, then $|f|$ is integrable on $[a,b]$. In a similar way, we have the following result:

Theorem 1 *If f and g are integrable on $[a,b]$, then the product fg is integrable on $[a,b]$.*

Proof. First notice that

$$fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right].$$

So, it is enough to prove that if f is integrable, then f^2 is integrable.

If f is integrable, then $|f|$ is integrable. Also, $f^2 = |f|^2$. Let $M = \sup|f|$ over $[a,b]$. Suppose that P is a partition of $[a,b]$, and $I_j = [x_{j-1}, x_j]$ for some j. Let

$$m_j = \inf_{x\in I_j} |f|\,, \quad M_j = \sup_{x\in I_j} |f|\,.$$

Observe that

$$m_j^2 = \inf_{x\in I_j} |f|^2\,, \quad M_j^2 = \sup_{x\in I_j} |f|^2\,.$$

But

$$M_j^2 - m_j^2 = (M_j + m_j)(M_j - m_j) \leq 2M(M_j - m_j)\,.$$

Therefore, for any partition P,

$$U\left(f^2, P\right) - L\left(f^2, P\right) \leq 2M\left(U\left(|f|, P\right) - L\left(|f|, P\right)\right).$$

This implies that f^2 is integrable, for any integrable f, and so fg is integrable for any integrable f and integrable g. ■

Now read: text: Substitution theorem (7.3.8), Integration by parts (7.3.17), Taylor's theorem with remainder (7.3.18)

1.0.1 Substitution

There are two ways to look at the method of substitution. The simplest is just as a way of finding antiderivatives. (Recall: F is an antiderivative of f if $F' = f$. This is often written as

$$\int f(t)\, dt = F(t) + c,$$

recognizing that if $F' = f$, then $(F + c)' = f$ as well, c being a constant function.)

Thus, we could do problem 16(d) in the text as follows:

Problem 2 *Let $f(t) = \frac{\cos\sqrt{t}}{\sqrt{t}}$ for $t > 0$. Find $\int_1^4 f$.*

Solution 3 *To use the fundamental theorem of calculus, we look for F such that $F' = f$. We have the formula*

$$\int F'(\phi(t))\, \phi'(t)\, dt = \int F'(u)\, du = F(u) = F(\phi(t))$$

if $u = \phi(t)$. Choosing $\phi(t) = \sqrt{t}$, we have $\phi'(t) = \frac{1}{2\sqrt{t}}$, and so $\int \frac{\cos\sqrt{t}}{\sqrt{t}} dt = \int \frac{2\cos\sqrt{t}}{2\sqrt{t}} dt = \int 2\cos u du = 2\sin u = 2\sin\left(\sqrt{t}\right)$. Thus, we can let $F(t) = 2\sin\left(\sqrt{t}\right)$, and we get the answer

$$\int_1^4 \frac{\cos\sqrt{t}}{\sqrt{t}} dt = 2\sin 2 - 2\sin 1.$$

But an alternate way of doing this is to change the variables in the limits of integration at the same time as we change variables in the integrand.[1]One way to think of this is as follows:

If $\int_a^b f(\phi(t))\, \phi'(t)\, dt = \int_c^d f(u)\, du$, what values of c and d make sense?

Answer: If $u = \phi(t)$, then when $t = a$, $u = \phi(a)$, and when $t = b$, $u = \phi(b)$. We thus get the formula of Theorem 7.3.8:

$$\int_a^b f(\phi(t))\, \phi'(t)\, dt = \int_{\phi(a)}^{\phi(b)} f(u)\, du.$$

[1] This is an important theoretical tool even when neither the integral nor the transformed integral can be evaluated exactly.

Thus, the solution to the problem above is:

$$\int_1^4 \frac{\cos\sqrt{t}}{\sqrt{t}}dt = \int_{\sqrt{1}}^{\sqrt{4}} 2\cos u du = 2\sin 2 - 2\sin 1.$$

But often substitutions are done differently. Consider

$$\int_0^1 \sqrt{1-t^2}dt.$$

We don't spot the form $f'(\phi(t))\phi'(t)$. Instead, we do the substitution backwards, in a sense. We let $t = \sin x$. This works because then $\sqrt{1-t^2} = \cos x$, if $0 \le t \le 1$ and $0 \le x \le \frac{\pi}{2}$. Now, we quietly calculate so that no one can see us: "$dt = \cos x\, dx$". This leads us to consider

$$\int_0^{\frac{\pi}{2}} \cos x \ \cos x \ dx.$$

We recognize this as an integral which can be done. To justify its relevance, we wish to write the integrand as

$$f(\phi(x))\phi'(x)$$

for some functions f and ϕ. Clearly, we need $\phi'(x) = \cos x$, so $\phi(x) = \sin x$ and $\sqrt{1-\phi(x)^2} = \cos x$. Hence by the previous substitution formula (using x in place of t, and t in place of u, we get

$$\int_a^b \cos x \ \cos x \ dx = \int_a^b \sqrt{1-\phi(x)^2}\phi'(x)\,dx = \int_{\phi(a)}^{\phi(b)} \sqrt{1-t^2}dt, \tag{1}$$

and we just have to determine a and b. We want $\phi(a) = 0$ and $\phi(b) = 1$, so we set $a = 0, b = \frac{\pi}{2}$. Thus,

$$\int_0^1 \sqrt{1-t^2}dt = \int_0^{\frac{\pi}{2}} \cos^2 x dx = \left(\frac{\sin 2x}{4} + \frac{x}{2}\right)\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

Question: What is the quick way of doing the original definite integral?

1.0.2 Integration by parts.

There is not much to say. Integration by parts is just a combination of the product rule for derivatives and the FTC. The product rule tells us that

$$(fg)' = f'g + g'f,$$

and so

$$\int_a^b (f'g + g'f) = \int_a^b (fg)' = f(b)\,g(b) - f(a)\,g(a)\,.$$

Hence,

$$\int_a^b f'g = f(b)\,g(b) - f(a)\,g(a) - \int_a^b g'f.$$

1.0.3 Taylor's Theorem with remainder.

There is a whole section on Taylor's theorem earlier (section 6.4), but we can get Taylor's formula using integration by parts. Suppose that f is a function such that f', f'',..., $f^{(n+1)}$ exist on $[a, b]$, and $f^{(n+1)}$ is integrable on $[a, b]$. Then consider the integral

$$\int_a^b f^{(n+1)}(t)\,(b-t)^n\,dt.$$

(We use dt notation when it is convenient!) Integrate this by parts.

$$\int_a^b f^{(n+1)}(t)\,(b-t)^n\,dt = f^{(n)}(t)\,(b-t)^n\Big|_a^b + \int_a^b nf^n(t)\,(b-t)^{n-1}\,dt.$$

One of the "boundary terms" vanishes, so we get

$$\int_a^b f^{(n+1)}(t)\,(b-t)^n\,dt = -f^{(n)}(a)\,(b-a)^n + \int_a^b nf^n(t)\,(b-t)^{n-1}\,dt.$$

Integrate by parts again and we get

$$\int_a^b f^{(n+1)}(t)\,(b-t)^n\,dt = -f^{(n)}(a)\,(b-a)^n - nf^{n-1}(a)\,(b-a)^{n-1} + \int_a^b n\,(n-1)\,f^{n-1}(t)\,(b-t)^{n-2}\,.$$

Keep doing this. The integral term after n integration by parts is

$$n! \int_a^b f'(t)\,dt = n!\,[f(b) - f(a)]\,.$$

So our final equation is

$$\int_a^b f^{(n+1)}(t)\,(b-t)^n\,dt = -f^{(n)}(a)\,(b-a)^n - nf^{n-1}(a)\,(b-a) - \cdots + n!\,(f(b) - f(a))\,.$$

We solve this for $f(b)$ and we get "Taylor's formula with remainder":

$$f(b) = f(a)+f'(a)(b-a)+\frac{1}{2}f''(a)(b-a)^2+\cdots+\frac{1}{n!}f^{(n)}(a)(b-a)^n+\frac{1}{n!}\int_a^b f^{(n+1)}(t)(b-t)^n\,dt,$$

The integral term at the end is called the remainder.

We can get another form of the remainder by assuming that f^{n+1} is continuous on $[a,b]$. Then let $m = \min_{a\le t\le b} f^{(n+1)}(t)$ and $M = \max_{a\le t\le b} f^{(n+1)}(t)$. Then the previous comparison theorem about integrals implies that

$$\int_a^b m(b-t)^n\,dt \le \int_a^b f^{(n+1)}(t)(b-t)^n\,dt \le \int_a^b M(b-t)^n\,dt.$$

Evaluating the first and last integrals implies that

$$m\frac{(b-a)^{n+1}}{n+1} \le \int_a^b f^{(n+1)}(t)(b-t)^n\,dt \le M\frac{(b-a)^{n+1}}{n+1}.$$

(This requires that $b-t\ge 0$; see below.) The intermediate value theorem implies that there must be some $d\in[m,M]$ with

$$\int_a^b f^{(n+1)}(t)(b-t)^n\,dt = d\frac{(b-a)^{n+1}}{n+1}.$$

(What function was used in the intermediate value theorem?) Since f^{n+1} is the derivative of $f^{(n)}$, it has the intermediate value property,[2] and so there must be some $c\in[a,b]$ with $f^n+1(c)=d$, giving

$$f(b) = f(a)+f'(a)(b-a)+\frac{1}{2}f''(a)(b-a)^2+\cdots+\frac{1}{n!}f^{(n)}(a)(b-a)^n+\frac{1}{(n+1)!}f^{(n+1)}(c)(b-a)^{n+1}.$$

This is the form of Taylor's theorem in Chapter 6. Of course, it is more common to write it as

$$f(x) = f(x_0)+f'(x_0)(x-x_0)+\cdots+R,$$

where R is the remainder term, written either as an integral or as a derivative.

[2]At the beginning I assumed that $f^{(n+1)}$ was continuous. But we only have to assume that $f^{(n+1)}$ exists in (a,b), and let m and M be inf and sup instead of max and min. This is a little more complicated, and perhaps the case of $f^{(n+1)}$ constant needs to be treated separately. But the result is still true because we have an intermediate value theorem for derivatives even if they are not continuous.

1.0.4 Intermediate value theorems for integrals

The following is easy to prove:

Theorem 4 *If f is continuous on $[a,b]$, then there is a $c \in (a,b)$ such that*

$$\int_a^b f = f(c)(b-a).$$

Proof. Since f is continuous, there are x_1 and x_1 in $[a,b]$ with $f(x_1) = \inf\{f(x) \ : \ x \in [a,b]\}$ and $f(x_2) = \sup\{f(x) \ : \ x \in [a,b]\}$. Then the comparison theorem for integrals implies that

$$f(x_1)(b-a) \leq \int_a^b f \leq f(x_2)(b-a).$$

The intermediate value theorem for continuous functions implies the theorem. ■

However, in the proof of Taylor's theorem with remainder, we used a slightly different result.

Theorem 5 *If f is continuous in $[a,b]$ and g is either nonnegative and integrable, or nonpositive. and integrable, then fg is integrable, and there is a $c \in (a,b)$ with*

$$\int_a^b fg = f(c)\int_a^b g. \tag{2}$$

We proved earlier that if f and g are integrable, then fg is integrable. We essentially gave the proof of the intermediate value formula in the previous section. What I want to emphasize here is the importance of the condition that g is either nonnegative or nonpositive. Suppose, for example, that $f(x) = e^x$ and $g(x) = x$, on $[-1,1]$. Then using integration by parts (or my latex program, which also does integrals!), we find that

$$\int_{-1}^1 fg = \int_{-1}^1 xe^x = \frac{2}{e}.$$

But $f > 0$, and $\int_{-1}^1 g = 0$, so the formula (2) can't be true here.