Math 0450
Honors intro to analysis
Spring, 2009
Notes 17

1 Uniform continuity

Read first: 5.4 Here are some examples of continuous functions, some of which we have done before.

1. $A = (0, 1]$, $f : A \to R$ given by $f(x) = \frac{1}{x}$.

   Proof. To prove that f is continuous at $c \in (0, 1]$, suppose that $\varepsilon > 0$, and let $\delta = \min\left\{\frac{c}{2}, \frac{c^2\varepsilon}{2}\right\}$. If $|x - c| < \delta$, then first of all, $x > \frac{c}{2}$ and so $0 < \frac{1}{x} < \frac{2}{c}$. Hence,
$$\left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{c - x}{xc}\right| = \left|\frac{1}{x}\right|\left|\frac{1}{c}\right||c - x| < \frac{2}{c}\frac{1}{c}\frac{c^2\varepsilon}{2} = \varepsilon.$$
   ■

2. $f : R \to R$ given by $f(x) = x^2$.

   Proof. Observe that
$$\left|x^2 - c^2\right| = |x - c|\,|(x + c)|.$$
   If $c = 0$, let $\delta = \sqrt{\varepsilon}$. If $c \neq 0$, let $\delta = \min\left\{|c|, \frac{\varepsilon}{2|c|}\right\}$ If $c = 0$ and $|x - c| < \delta$, then $|x| < \sqrt{\varepsilon}$ and $|x^2 - c^2| = |x^2| < \varepsilon$, If $c \neq 0$, then
$$\left|x^2 - c^2\right| = |x + c|\,|x - c| < 2\,|c|\,\frac{\varepsilon}{2\,|c|} = \varepsilon.$$
   ■

3. $A = [1, \infty)$, $f(x) = \frac{1}{x}$.

   Proof. This makes use of item 1 above. Suppose that $c \in A$. From item 1, we see that if $\varepsilon > 0$, and $|x - c| < \min\left\{\frac{c}{2}, \frac{c^2\varepsilon}{2}\right\}$, then $\left|\frac{1}{x} - \frac{1}{c}\right| < \varepsilon$. Hence, let $\delta = \frac{\varepsilon}{2}$. If $|x - c| < \delta$, and x and c are in A, then $|x - c| < \min\left\{\frac{c}{2}, \frac{c^2\varepsilon}{2}\right\}$, so $\left|\frac{1}{x} - \frac{1}{c}\right| < \varepsilon$. ■

4. $A = [1, 2]$, $f(x) = \frac{1}{x}$.

   Proof. Any δ which works in #3 clearly works here. So if $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{2}$. This brings out the point that if f is uniformly continuous on a set A, and $B \subset A$, then f is also uniformly continuous on B. ■

Now we point out an important difference between examples 1 and 2 and examples 3 and 4. In 1 and 2, the δ depends on c . In 3 and 4 it does not.

Definition 1 *Suppose $f : A \to R$. We say that f is "uniformly continuous" on A if for each $\varepsilon > 0$ there is a $\delta > 0$ such that if c and x are in A, and $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.*

In this definition it is very important that δ is chosen before c, so that δ does not depend on c. Looking at the examples above, we see that:

1. $\frac{1}{x}$ is not uniformly continuous on $(0, 1]$
2. x^2 is not uniformly continuous on $[0, \infty)$
3. $\frac{1}{x}$ is uniformly continuous on $[1, \infty)$
4. x^2 is uniformly continuous on $[1, 2]$.

One further example:

Proposition 2 *$\sqrt{x}$ is uniformly continuous on $[0, \infty)$.*

Proof. Suppose that $\varepsilon > 0$. Let $\varepsilon_1 = \min\{\varepsilon, 1\}$. We consider two cases: (i) $0 \le c < \frac{\varepsilon_1^2}{9}$ and (ii) $\frac{\varepsilon_1^2}{9} \le c$. In case (i), let $\delta = \frac{\varepsilon_1^2}{9}$. If $|x - c| < \delta$ and $x \ge 0$ then $0 \le x < c + \frac{\varepsilon_1^2}{9} \le 2\frac{\varepsilon_1^2}{9}$. Hence,

$$\left|\sqrt{x} - \sqrt{c}\right| \le \sqrt{x} + \sqrt{c} < \sqrt{\frac{2}{9}}\varepsilon_1 + \frac{1}{3}\varepsilon_1 < \varepsilon_1 \le \varepsilon.$$

In case (ii), observe that

$$\left|\sqrt{x} - \sqrt{c}\right| = \left|\sqrt{x} - \sqrt{c}\right| \left|\frac{\sqrt{x} + \sqrt{c}}{\sqrt{x} + \sqrt{c}}\right| = \left|\frac{x - c}{\sqrt{x} + \sqrt{c}}\right| \le \frac{|x - c|}{\sqrt{c}} \le \frac{|x - c|}{\varepsilon_1 / 3}.$$

Hence, we choose $\delta = \frac{\varepsilon_1}{3}\varepsilon$.

If we let $\delta(\varepsilon) = \min\left\{\frac{\varepsilon_1^2}{9}, \frac{\varepsilon_1}{3}\varepsilon\right\}$, then this works in both cases, and shows that f is uniformly continuous on $[0,\infty)$. ■

Now we have an important theorem

Theorem 3 *If $A = [a,b]$ is a closed bounded interval, and $f : A \to R$ is continuous on A, then f is uniformly continuous on A.*

Proof. *If not, then there is an $\varepsilon > 0$ such that for each $\delta > 0$, there is a pair $\{c, x\}$ with $|x - c| < \delta$ and $|f(x) - f(c)| \geq \varepsilon$. In particular, this is true if $\delta = \frac{1}{n}$ where $n \in \mathcal{N}$. So we obtain two sequences, $(c_n) \subseteq [a,b]$ and $(x_n) \subseteq [a,b]$, such that $|x_n - c_n| < \frac{1}{n}$ and $|f(c_n) - f(x_n)| \geq \varepsilon$. But by the Bolzano-Weierstrass theorem, (c_n) must have a convergent subsequence (c_{n_k}). Suppose that (c_{n_k}) converges to c. Since $|x_{n_k} - c_{n_k}| < \frac{1}{n_k}$, it must be the case that $(x_{n_k}) \to c$ as well. But f is continuous at c, so that $\lim(f(x_{n_k})) = f(c)$ and $\lim(f(c_{n_k})) = f(c)$. In other words,*

$$\lim(f(x_{n_k}) - f(c)) = 0$$
$$\lim(f(c_{n_k}) - f(c)) = 0.$$

Since

$$|f(c_{n_k}) - f(x_{n_k})| = |f(c_{n_k}) - f(c) + f(c) - f(x_{n_k})| \leq |f(c_{n_k}) - f(c)| + |f(x_{n_k}) - f(c)|,$$

it follows that $\lim(f(c_{n_k}) - f(x_{n_k})) = 0$. Since $|f(c_n) - f(x_n)| \geq \varepsilon$ for all n, this is a contradiction. ■

Once again, the Bolzano-Weierstrass theorem plays a key role.

Example 4 *Now we can easily show that $\sqrt{x}$ is uniformly continuous. It was shown in Theorem 5.2.5 that $\sqrt{x}$ is continuous at each $c \geq 0$. In particular, it is continuous on $[0,2]$, and so uniformly continuous. So for each $\varepsilon > 0$, there is a δ_1 independent of $c \in [0,2]$ such that if $|x - c| < \delta$, then $|\sqrt{x} - \sqrt{c}| < \varepsilon$. Let $\delta = \min\{\delta_1, \varepsilon, 1\}$. If $c \geq 1$ and $|x - c| < \delta$, then (as in (iii) in our earlier discussion of $\sqrt{x}$)*

$$|\sqrt{x} - \sqrt{c}| = \left|\frac{x - c}{\sqrt{x} + \sqrt{c}}\right| \leq |x - c| < \varepsilon.$$

(We could chose the intervals $[0,1]$ and $[1,\infty)$, but this leads to unnecessary complication when c is at or close to 1.) Hence, for any $c \geq 0$, if $|x - c| < \delta$, then $|\sqrt{x} - \sqrt{c}| < \varepsilon$.

Lipschitz continuity

Definition 5 *A function $f : A \to R$ is said to be "Lipschitz continuous" on A if there is an $L \in R$ such that for each $x \in A$ and $y \in A$, $|f(x) - f(y)| \leq L|x - y|$.*

It is easily shown that if f is Lipschitz continuous on A, then f is uniformly continuous on A. Just choose $\delta = \frac{\varepsilon}{L}$:

$$|f(x) - f(c)| \leq L|x - y| < L\frac{\varepsilon}{L} = \varepsilon.$$

As an example we have $f(x) = x$ on R. Even though R is unbounded, f is uniformly continuous on R. f is Lipschitz continuous on R, with $L = 1$. This shows that if A is unbounded, then f can be unbounded and still uniformly continuous.

The function x^2 is an easy example of a function which is continuous, but not uniformly continuous, on R.

If we jump ahead, and assume we know about derivatives, we can see a relation between $f'(x)$ and Lipschitz continuity. Eventually we will prove that if f is differentiable on $[a, b]$, then there is a $c \in [a, b]$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

Hence, if the function f' is bounded on $[a, b]$, and L is an upper bound for $|f'(x)|$ on $[a, b]$, then

$$|f(b) - f(a)| \leq L|b - a|$$

Example 6 *$f(x) = x^2$ on $I = [0, 100]$. Since $|f'(x)| = |2x| \leq 200$ on I, we see that x^2 is Lipschitz continuous, and so uniformly continuous, on I.*

Example 7 *$f(x) = x^2 = [0, \infty) = \{x \in R \ : \ x \geq 0\}$. The derivative $f'(x) = 2x$ is unbounded on on $[0, \infty)$, and f is not Lipschitz continuous there. We saw earlier that x^2 is also not uniformly continuous on $[0, \infty)$.*

Example 8 *$f(x) = \sqrt{x}$ on $I = (0, \infty)$. Now $f'(x) = \frac{1}{2\sqrt{x}}$, which is unbounded on I. Hence, f is not Lipschitz continuous on I. However, we saw above that $\sqrt{x}$ **is** uniformly continuous on $(0, \infty)$.*

1.1 Approximation of functions

Often it is desirable to approximate a function f by a simpler function g. Among the so-called "simpler" functions which may be used are the "piecewise constant" functions and the "piecewise linear" functions. To define these, we first define a "partition" of an interval, something which will be very useful later on.

Definition 9 *Suppose that* $I = [a,b]$ *is a closed bounded interval. Then a "partition" of* I *is a finite set of points* $\{x_0, x_1, .., x_n\}$ *with* $a = x_0 < x_1 < \cdots < x_n = b$.

Definition 10 *A function* g *is said to be "piecewise constant" on an interval* $[a,b]$ *if there is a partition* $\{x_0, ..., x_n\}$ *such that* g *is constant on* (x_i, x_{i+1}) *and the values at the partition points are the limits from one side or the other. Such functions are also called "'step" functions.*

The following theorem says that a continuous function on a closed bounded interval can be approximated by step functions.

Theorem 11 *Suppose that* $f : [a,b] \to R$ *and* f *is continuous in* $[a,b]$. *Then for each* $\varepsilon > 0$ *there is a piecewise linear function* g_ε *such that*

$$|f(x) - g_\varepsilon(x)| < \varepsilon$$

for all $x \in [a,b]$.

The proof is in the text, and relies on the uniform continuity of f.

Definition 12 *A function* g *is said to be "piecewise linear"' if there is a partition* $\{x_0, ..., x_n\}$ *such that* g *is a linear function* $(ax+b)$ *on* (x_i, x_{i+1}), *and the values at the partition points are the limits from one side or the other.*

A piecewise linear function does not have to be continuous.

Theorem 13 *A continuous function on a closed bounded interval can be approximated by a continuous piecewise linear function on that interval.*

Again, the proof is in the text.

Another useful set of functions to use in approximations is the set of polynomial functions

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

If x_0 is any constant, then

$$a_n (x - x_0)^n + a_{n-1} (x - x_0)^{n-1} + \cdots + a_1 (x - x_0) + a_0$$

is also a polynomial, and can be put into the form of $p(x)$ just above by using the binomial theorem.

Again jumping ahead to calculus, it is shown there that many functions can be approximated by their "Taylor polynomials". If f, f', f'', and indeed all the derivatives of f exist on an interval I, and $c \in I$, then the Taylor polynomial of order n around c is the polynomial

$$a_0 + a_1 (x - x_0) + \cdots + a_n (x - x_0)^n$$

if

$$a_i = \frac{f^{(i)}(c)}{i!}.$$

Here $f^{(i)}$ denotes the i^{th} derivative of f.

However, not all functions can be approximated by their Taylor polynomials. This is certainly true if the function is not differentiable everywhere. But even a very "smooth" function might not be close to its Taylor polynomials. As an example, let $f : [0, \infty) \to R$ be defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x}} \text{if } x > 0 \\ 0 \text{ if } x = 0. \end{cases}$$

Note that $\lim_{x \to 0^+} f(x) = 0$. Also, $f'(x) = e^{-\frac{1}{x}} \left(-\frac{1}{x^2}\right)$if $x \neq 0$, from which it follows (as we will see later) , that $\lim_{x \to 0^+} f'(x) = 0$. In fact, every zero approaches 0 as $x \to 0^+$.

This is not the same as saying that $f'(0) = 0$, $f''(0) = 0$, and so forth, but we will also show later that $f^{(j)}(0) = 0$ for every $j > 0$ in $\mathcal{N}$. Hence, for every n the Taylor polynomial of $e^{-\frac{1}{x}}$ around 0 of order n is the 0 polynomial. This is obviously not a good approximation of $e^{-\frac{1}{x}}$.

However, it turns out that there is a family of polynomials which will approximate f. This is a consequence of the Weierstrass approximation theorem:

Theorem 14 *Suppose that* $f : [a, b] \to R$ *is continuous. Then for any* $\varepsilon > 0$, *there is a polynomial* $p_\varepsilon(x)$ *such that*

$$[f(x) - p_\varepsilon(x)] < \varepsilon$$

for $a \le x \le b$.

The proof is outlined in the text, but full details are not given. Unfortunately, we shall not have time to give this proof either.

2 Monotone functions and inverses

Read first: Section 5.6. (We will skip 5.5.) The terminology here is a little confusing. In particular, the statement that a function is "increasing" is ambiguous in the literature. Following the text, we say that f is "increasing" if whenever $x_1 > x_2$, we have $f(x_1) \ge f(x_2)$. We say that f is "strictly increasing" if, whenever $x_1 > x_2$, we have $f(x_1) > f(x_2)$.

This section is fairly easy. When a function is strictly increasing, or strictly decreasing, it has an inverse. It pays, though, to recall the definition of a function and its inverse. See the first set of notes, and pages 8-9 of the text. There is one less obvious theorem. In this theorem, I is an interval, which may be open, closed, or neither.

Theorem 15 *If* $f : I \to R$ *is strictly increasing and continuous, then its inverse is strictly increasing and continuous.*

Proof. We first show that f is strictly strictly increasing on I. Suppose that $y_1 > y_2$ and that y_1 and y_2 are both in the range $f(I)$. Suppose, however, that if $x_1 = f^{-1}(y_1)$ and $x_1 = f^{-1}(y_2)$, then $x_1 < x_2$. By the strict monotonicity of f, then $y_1 = f(x_1) < y_2 = f(x_2)$,a contradiction.

Now suppose that f^{-1} is discontinuous at some point $d \in f(I)$. Recall that $f(I)$ is an interval (Theorem 5.3.10). Assume that d is not an endpoint of $f(I)$. Since f^{-1} is discontinuous,

$$x_1 = \lim_{y \to d^-} f^{-1}(y) = \sup\left\{f^{-1}(y) \ : \ y < d\right\} < \lim_{y \to d^+} f^{-1}(y) = \inf\left\{f^{-1}(y) \ : \ y > d\right\} = x_2.$$

Hence, $f(x_1) < f(x_2)$. Furthermore, there is no $y < d$ with $f^{-1}(y) > x_1$ and no $y > d$ with $f^{-1}(y) < x_2$.

Choose a $d_1 \in (f(x_1), f(x_2))$ with $d_1 \neq d$. (There must be such a d_1, since d is only one point, but $(f(x_1), f(x_2))$ is an interval.) By the intermediate value theorem, there is a $c_1 \in (x_1, x_2)$ with $f(c_1) = d_1$. But then, $f^{-1}(d_1) = c_1 \in (x_1, x_2)$. This contradicts the last sentence of the previous paragraph. ■

3 Homework Due March 26

pg. 144, #2, 7 (The notes above may be helpful. You can use facts from calculus about $\sin x$.)

pg. 144, # 13

pg. 155, #8,9,13

pg. 166, # 1(d)