Math 0450
Honors intro to analysis
Spring, 2009
Notes 16

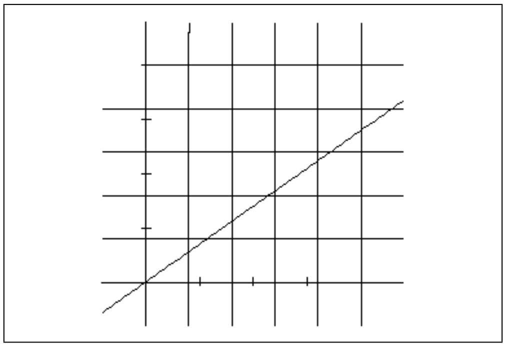
SIMPLER PROOF OF PROBLEM 1 FROM LAST WEEK'S HOMEWORK This was the problem of showing that the Cantor set $\mathcal{F}$ is closed. A few students gave a much simpler proof than the one I gave previously. It is not necessary to use the ternary expansion.. (So the hint was not necessary.)
Proof. Suppose that $x \in [0,1]$ but $x \notin \mathcal{F}$. Then from the definition of $\mathcal{F}$, x is in one of the open intervals that was removed from $[0,1]$ in defining $\mathcal{F}$. If (a,b) is one of those intervals, $x \in (a,b)$, $\varepsilon = \min\{x-a, b-x\}$, and c is any element of $\mathcal{F}$, then $|c-x| \geq \varepsilon$. Hence, x cannot be a limit point of a sequence in $\mathcal{F}$. ■

A second proof uses an extension of the nested closed interval result This extension is 11.1.5 on page 314, and says that any set which is the intersection of a family of closed sets is also a closed set. The proof uses deMorgan's laws from Chapter 1. However, we did not cover chapter 11, so I took off a couple of points for this proof. The proof in the first paragraph is ok, however.

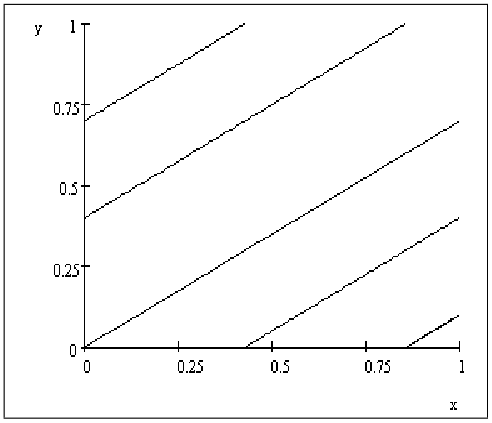
An interesting way to think about problems 2 and 3. Suppose that $\alpha \in R$, and consider the graph of the function

$$f(x) = \alpha x.$$

Imagine the graph done on a grid showing all the horizontal and vertical lines $x = n, y = m$.



From this graph, we construct a graph which is entirely in the region $0 \leq x \leq 1, 0 \leq y \leq 1$ by superimposing any of the other squares through which the line passes on top of this one. This gives a set of lines, of which only the first few are shown below.



Every time one of these lines hits the right edge $x = 1$, we get a new point in the set $\{n\alpha + m \ : \ n, m \in Z\}$. The problem was to show that if α is irrational, then eventually, these lines fill up more and more of the square, leaving smaller and smaller gaps. In particular, the set of points on the right edge fills in the right edge, with smaller and smaller gaps.

This doesn't prove the result, however. The Bolzano-Weierstrass theorem is still the key, along with the theorem that a sequence which converges is a Cauchy sequence.

We can carry this one step further. Imagine that the unit square above is a piece of paper, and wrap it around a rod of circumference 1, so as to make a cylinder. Then the lines becomes pieces of helices going around the cylinder. Now imagine that the cylinder is made out of rubber, and the top and bottom are fit together to make a bagel or doughnut shape – a "torus" in mathematical terminology. Then the pieces of helices will match up to form a single curve wrapping around the torus many times. The theorem then is that this curve, while not passing through every point, will come so close to every point that it will look like the whole torus is filled up. Again, this doesn't help the proof; it just makes a nice way of picturing the result.

Remark about the definition of limit. We gave two definition in notes 14, Definition 1 and Definition 6. Definition 1 was simpler, because $f : (a, b) \to R$, where (a, b) is an open interval, and $x \in (a, b)$. In Definition 6, and in Definition 4.1.4 in the text, there was the complication that $f : A \to R$, where A is any subset of R, and c had to be a cluster point of A.

One setting where this added feature is important is still relatively simple. For example, we could have $f : (0,1) \to R$ be defined by

$$f(x) = \frac{\sin x}{x}.$$

When we discuss trig functions, we will see that

$$\lim_{x\to 0} f(x) = 1.$$

This uses the "cluster point" idea, because $0 \notin (0,1)$, but 0 is a cluster point of $(0,1)$, and so the limit is defined. This should not be hard to understand.

A more complicated situation is when A is a more complicated set. For example, A could be the set of rational numbers. This is when the more complicated definitions are most needed.

1 Continuous functions

Read first: section 5.1 Suppose that $A \subseteq R$ and $f : A \to R$. Further, suppose that $c \in A$.

Definition 1 *The function f is said to be continuous at c if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $x \in A$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.*

Notice the differences from the definition of limit.

(a) In this definition, $c \in A$, so that $f(c)$ is defined.

(b) There is no requirement that c be a cluster point of A. But the definition is not very useful if c is not a cluster point of A. You can show that if c is not a cluster point, then every function defined on A is continuous at c.

(c) It is not necessary to specify that $0 < |x - c|$, since if $0 = |x - c|$, then $f(x) = f(c)$, and so $|f(x) - f(c)| = 0 < \varepsilon$ for any $\varepsilon > 0$.

Remark 2 *The definition says that f is continuous "at c". This is not a definition of a "continuous function". It is a definition of what it means to say that f is continuous at a particular point in its domain.*

Example 3 *Let*

$$f(x) = \begin{matrix} 1 \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational.} \end{matrix}$$

Then f *is not continuous at any* c.

Example 4 *Let*

$$g(x) = \begin{matrix} x \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational.} \end{matrix}$$

Then g *is continuous at* 0, *but if* $c \neq 0$, *then* g *is discontinuous at* c.

Example 5 *Let*

$$h(x) = \begin{matrix} x \text{ if } x \neq 0 \\ 1 \text{ if } x = 0. \end{matrix}$$

Then h *is discontinuous at* 0, *but if* $c \neq 0$, *then* h *is continuous at* c.

Example 6 *Let*

$$p(x) = \begin{matrix} \frac{1}{q} \text{ if } x = \frac{p}{q} \text{ and } p, q \in Z \text{ have no common integer factors except } 1 \\ 0 \text{ if } x \text{ is irrational.} \end{matrix}$$

If c *is irrational, then* p *is continuous at* c, *while if* c *is rational, then* p *is discontinuous at* c.

This is proved in the text, Example 5.1.6(h).

Remark 7 *It turns out that it is impossible to define a function which is continuous at each rational number and discontinuous at each irrational number. This is quite a "deep" result. (Mathematicians love to use the word "deep". It seems to imply something that is far from obvious and requires a lot of background to discover or to understand.)*

Definition 8 *If* $f : A \to R$ *and* f *is continuous at every* $c \in A$, *then* f *is said to be "continuous on* A*".*

Here are some important theorems about continuity:

Theorem 9 *If $f : A \to R$, $c \in A$, and c is a cluster point of A, then f is continuous at c if and only if $c \in A$ and*

$$\lim_{x \to c} f(x) = f(c).$$

Implicit in this definition is that $\lim_{x \to c} f(x)$ exists.

Theorem 10 *Under the same hypotheses as the previous theorem, f is continuous at c if and only if for every sequence $(x_n) \subseteq A$ such that* $\lim (x_n) = c$, *it is the case that* $\lim (f(x_n)) = f(c)$.

Removable discontinuities As pointed out in the text, sometimes a function is discontinuous at a point c because $c \notin A$, so that $f(c)$ is not defined. But suppose that c is a cluster point of A, and suppose that $\lim_{x \to c} f(x)$ exists. Say

$$\lim_{x \to c} f(x) = L.$$

Then we can define a new function, say f_1, as follows:

$$f_1(x) = \begin{matrix} f(x) \text{ if } x \in A \\ L \text{ if } x = c \end{matrix}$$

Then f_1 is defined on $A \cup \{c\}$, and since

$$\lim_{x \to c} f_1(x) = f_1(c),$$

the function f_1 is continuous. We call f_1 the "continuous extension" of f to $A \cup \{c\}$. For example,

$$f_1(x) = \begin{matrix} \frac{\sin x}{x} \text{ if } x \neq 0 \\ 1 \text{ if } x = 0 \end{matrix}$$

is continuous on R. (This is proved in Chapter eight.)

The "greatest integer" function This is a useful example of a discontinuous function. For each $x \in R$, let

$$[[x]] = \sup \{ n \in \mathcal{N} \ : \ n \leq x \} .$$

Note that this exists, since $\{ n \in \mathcal{N} \ : \ n \leq x \}$ is a bounded set. (x is an upper bound.)

We see that

$$\lim_{x \to 0^-} [[x]] = -1$$
$$\lim_{x \to 0^+} [[x]] = 0,$$

and since these two limits are different, $\lim_{x \to 0} [[x]]$ does not exist, so $[[]]$ is discontinuous at 0. It is also discontinuous at each $n \in Z$.

This function would have been useful on the last homework assignment. Notice that for any x,

$$x - [[x]] \in [0, 1).$$

Also, $[[x]]$ is always an integer. Hence problem 3 could be stated as follows:

Suppose that α is irrational. Prove that the set

$$\{ n\alpha - [[n\alpha]] \ : \ n \in Z \}$$

is a dense subset of $[0, 1]$.

2 Theorems about continuous functions.

Read: 5.2,5.3. The theorems in 5.2 should sound familiar. They are just like the theorems in 3.2 and 4.2. For example, if f and g are continuous, then $f + g$ is continuous. Most of these are proved using the equivalence of convergence of limits by the definition and convergence by considering sequences. As a result of this section, all the usual functions are continuous, so long as we are not dividing by zero.

Section 5.3 has some important general theorems. Several of these illustrate the importance of the Bolzano-Weierstrass theorem, since that is used in their proof.

The so-called “maximum-minimum theorem” could use some explanation. This theorem emphasizes the difference between the “maximum” of a bounded function on a set and the “supremum” of a bounded function on a set.

Suppose that $A \subseteq R$ and $f : A \to R$. Recall that f is “bounded”, or “bounded on A”, if there is a Y such that

$$f(x) \leq Y$$

for all $x \in A$. In this case, the image set, or range, $f(A)$ is bounded above, with Y being an upper bound. By the completeness axiom for R,

$$s = \sup f(A)$$

exists.

But this does not mean that there is any $x \in A$ with $f(x) = s$. As a simple example, let $A = (0, 1)$ and $f(x) = \frac{x}{x^2+1}$. Then you can easily show that

$$\sup f(A) = \frac{1}{2}.$$

However there is no $x \in (0, 1)$ such that $f(x) = \frac{1}{2}$.

This is a rather artificial example, since we can obviously extend f to a function $f_1 : B \to R$, where $B = [0, 1]$. The formula is the same: $f_1(x) = 2x$. In this case,

$$\sup f_1 = \frac{1}{2},$$

and

$$f(1) = \frac{1}{2}.$$

So we say that the supremum is actually “taken on”, and we call $\frac{1}{2}$ the “maximum” of f_1 on B. The text uses the term “absolute maximum” for this quantity.

As an example of a function which definitely has only a supremum, and not a maximum, consider $g : R \to R$ given by

$$g(x) = \frac{x^2}{x^2 + 1}.$$

Then

$$\sup g(R) = 1,$$

but there is no $x \in R$ with $g(x) = 1$. So the “maximum” of g does not exist, while its supremum is 1.

The maximum-minimum theorem is Theorem 5.3.4, and states that if A is a closed bounded interval (or it could be a closed bounded set, such as $[0,1] \cup [2,3]$), and if $f : A \rightarrow R$ is continuous on A, then f has a maximum and a minimum on A. The proof uses the Bolzano-Weierstrass theorem.

Remark 11 *The interval* $R = (-\infty, \infty)$ *is a closed interval, since every limit point of a sequence in this interval is in the interval. But we saw that the conclusion of the maximum-minimum theorem is not true for this interval. This is why the hypotheses say that* A *must be bounded, as well as closed.*

2.1 A practical application of the theory

Most of what we have done is quite abstract, so it may be surprising that it has practical consequences. Often in applied mathematics it is important to solve an equation of the form

$$f(x) = 0.$$

For example, we might want to solve

$$f(x) = x^5 - 3x^4 + 3x^3 - 2 = 0.$$

There is no general formula for such a solution, and we may wonder if there even is a solution.[1] So the "location of roots" theorem, Theorem 5.3.5, has practical importance. You can read this in class, but for this example we proceed as follows:

Observe that $f(0) < 0$ and $f(2) > 0$. We define a sequence of closed intervals $[a_n, b_n]$ as follows. Let

$$I_1 = [a_1, b_1] = [0, 2].$$

Calculate that $f(1) < 0$. As a result, we let

$$I_2 = [a_2, b_2] = [1, 2],$$

so that

$$f(a_2) < 0, f(b_2) > 0. \tag{1}$$

Now suppose that $[a_1, b_1], ..., [a_k, b_k]$ have been chosen, with $f(a_k) < 0$ and $f(b_k) > 0$, and with

$$|b_k - a_k| = \left(\frac{1}{2}\right)^{k-1} |a_1 - b_1|. \tag{2}$$

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[1] For example, $x^2 + 1 = 0$ has no solution in R.

Let $c_k = \frac{a_k+b_k}{2}$. If $f(c_k) = 0$ we are done. If $f(c_k) < 0$, let

$$I_{k+1} = [a_{k+1}, b_{k+1}] = [c_k, b_k].$$

If $f(c_k) > 0$, let

$$I_{k+1} = [a_{k+1}, b_{k+1}] = [a_k, c_k].$$

In either case (1) and (2) hold with $k+1$ in place of k .In this way we form a sequence of nested closed bounded intervals, and the set

$$S = \cap_{n=1}^{\infty} I_n$$

is nonempty. Because $\lim(b_n - a_n) = 0$, we can prove that S contains a single point, say x^*. Also, the construction shows that (a_n) is an increasing sequence and that $\lim(a_n) = x^*$. Because f is continuous (by the theorems in 5.2), we must have $\lim(f(a_n)) = f(x^*)$. Since $f(a_n) < 0$, it follows that $f(x^*) \leq 0$. Also, $\lim(b_n) = x^*$ and $f(b_n) > 0$, so $f(x^*) \geq 0$. Hence, $f(x^*) = 0$.

The practical aspect is that this gives a way of finding roots of an equation, once we find a_1 and b_1 with $f(a_1) < 0$, $f(b_1) > 0$. For the example above,

$$\begin{aligned}
I_2 &= [1,2] \text{ and } f(1.5) = 0.531\,25..\\
I_3 &= [1,1.5] \text{ and } f(1.25) = -0.413\,09..\\
I_3 &= [1.25,1.5] \text{ and } f(1.375) = -0.009641..\\
I_4 &= [1.375,1.5] \text{ and } f(1.4375) = .23945..
\end{aligned}$$

and so forth. We have shown that there is a root between 1.375 and 1.4375. (We can guess that it is much closer to 1.375, based on the numbers above.)

But if one wants to do this millions of times, it would pay to find a faster method. Later we may have time to discuss “Newton’s method”, which indeed gives much faster convergence. However, it is often necessary to use this “bisection” method to get close enough to a solution for Newton’s method to work.

3 Homework due March 17

1,2:pg. 118, #5 (a,f)

3. The current postage rate for a first class letter is as follows:

weight not over 1 ounce: $.42
weight not over 2 ounces: $.59
weight not over 3 ounces: $.76
weight not over 3.5 ounces: $.93

Express this in terms of a single formula using the greatest integer function. Hint: Should your function be “continuous from the left ” or “continuous from the right”?

4,5: pg.124, # 11,12

6,7,8: pg. 129, # 5,8,15

9,10: pg. 135, # 9, 12.