Math 0450
Honors intro to analysis
Spring, 2009
Notes 15

1 Some further examples of limits

In these examples I use the definition of limit, rather than the theorems in section 4.2. Also, in these examples, I will stop referring to "$\delta(\varepsilon)$", and just write "δ", with the dependence of δ on ε understood.

Example 1

$$\lim_{x\to 2}\frac{1}{x}=\frac{1}{2}.$$

Proof. Observe that if $x \neq 0$, then

$$\left|\frac{1}{x}-\frac{1}{2}\right|=\left|\frac{2-x}{2x}\right|=\frac{1}{2}\frac{1}{|x|}\,|2-x|\,.$$

Suppose that $\varepsilon > 0$.

Remark 2 *We wish to have* $\left|\frac{1}{x}-\frac{1}{2}\right|<\varepsilon$. *This means that in the line above, the term* $\frac{1}{|x|}$ *cannot be too large. In other words, we need an* ***upper bound*** *for this term. To keep* $\left|\frac{1}{x}\right|$ *from being too large, we need to keep* $|x|$ *from being too small. Since we are taking the limit as* $x\to 2$, *we will make an assumption which insures that* $x > 1$.

Let $\delta = \min\{1,\varepsilon\}$. If $|x-2| < \delta$, then in particular, $x > 1$, and so $\left|\frac{1}{x}\right| < 1$. Hence

$$\left|\frac{1}{x}-\frac{1}{2}\right|<\frac{1}{2}\,|2-x|<\frac{1}{2}\varepsilon<\varepsilon.$$

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Example 3 *Prove that if* $c \neq 0$, *then*

$$\lim_{x\to c}\frac{1}{x}=\frac{1}{c}.$$

Proof. This is to illustrate how to turn the special case of the previous example, where $c = 2$ into a more general result. Having worked out the previous example, this one becomes easier. One tricky point is how to handle the possibility that $c < 0$. So to start with, I will assume that $c > 0$.

Observe that

$$\left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{c-x}{cx}\right| = \left|\frac{1}{c}\right|\left|\frac{1}{x}\right||c-x|. \tag{1}$$

Suppose that $\varepsilon > 0$. Assume first that $\delta < \frac{c}{2}$. If $|x-c| < \delta$, then in particular, because $c > 0$, it follows that $|x - c| < \frac{c}{2}$, so

$$c - \frac{c}{2} < x < c + \frac{c}{2}. \tag{2}$$

Hence, $x > \frac{c}{2}$ and $\left|\frac{1}{x}\right| < \frac{2}{c}$. Therefore, from (1),

$$\left|\frac{1}{x} - \frac{1}{c}\right| < \frac{2}{c^2}|c-x|.$$

Therefore, let $\delta = \min\left\{\frac{c}{2}, \frac{c^2}{2}\varepsilon\right\}$. If $|x-c| < \delta$, then

$$\left|\frac{1}{x} - \frac{1}{c}\right| < \frac{2}{c^2}|c-x| < \varepsilon.$$

If $c < 0$, then $|x - c| = |(-x) - (-c)|$ and

$$\left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{1}{-x} - \frac{1}{-c}\right|. \tag{3}$$

Suppose that $\varepsilon > 0$. Let $\delta = \min\left\{\frac{|c|}{2}, \frac{c^2}{2}\varepsilon\right\}$. If $|x-c| < \delta$, then $|(-x)-(-c)| < \delta$ and so the previous calculations show that

$$\left|\frac{1}{x} - \frac{1}{c}\right| < \varepsilon.$$

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Remark 4 *The important difference between proving that*

$$\lim_{x\to 2}\frac{1}{x} = \frac{1}{2}$$

and proving that

$$\lim_{x \to c} \frac{1}{x} = \frac{1}{c}$$

is that in the second case, δ must depend on c as well as on ε. Also, the case of $c < 0$ must be covered. If you are faced with a problem like this, and don't see how to do it, try substituting a specific number for c first.

Example 5 *Prove that*

$$\lim_{x \to -3} |x| = 3.$$

Proof. Suppose that $\varepsilon > 0$. Let $\delta = \min\{3, \varepsilon\}$. If $|x - (-3)| < \delta$, then in particular, $x < 0$, and so $|x| = -x$. Hence,

$$||x| - 3| = |-x - 3| = |x + 3| = |x - (-3)| < \varepsilon.$$

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1.1 Extended definition of limit

Read: Section 4.3.

Example 6 *Suppose that $A = Q$, and $f : A \to R$ is defined by*

$$f(x) = 2^{\frac{1}{x}} \text{ for } x \in Q.$$

Prove that

$$\lim_{x \to 0^-} f(x) = 0.$$

Proof. Suppose that $\varepsilon > 0$. Let $\varepsilon_1 = \min\left\{\varepsilon, \frac{1}{2}\right\}$. Observe that for any $x < 0$ in Q, $2^{\frac{1}{x}} < \varepsilon_1$ if and only if $\left(\frac{1}{2}\right)^{\frac{1}{-x}} < \varepsilon_1$, and this is true if and only if $\frac{1}{2} < \varepsilon_1^{-x}$. In example 3.1.11(c) it was shown that for any $c > 0$, $\lim\left(c^{\frac{1}{n}}\right) = 1$. Hence , there is an n_1 such that if $n \geq n_1$, then $\varepsilon_1^{\frac{1}{n_1}} > \frac{3}{4}$. Let $\delta = -\frac{1}{n_1}$. If $x < 0$ and $|x - 0| < \delta$, then $0 < -x < \frac{1}{n_1}$, which implies that

$$\frac{1}{2} < \varepsilon_1^{\frac{1}{n_1}} < \varepsilon_1^{-x}.$$

This is is because ε_1^y is a decreasing function of y if $y > 0$ and $0 < \varepsilon_1 < 1$. Hence, for $x < 0$, $x \in Q$, and $|x - 0| < \delta$,

$$|f(x) - 0| < \varepsilon_1 \leq \varepsilon.$$

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Remark 7 *In the text it is shown that* $\lim_{x \to 0^-} e^{\frac{1}{x}} = 0$. *Use is made of an inequality from Chapter 8. That is reasonable, but above, I tried to use only what has come earlier. We haven't defined* a^x *for an arbitrary* x, *only for rational numbers. And while we have defined* e, *it seemed more elementary to use* 2.

2 Homework, due Thursday, March 5

1. Definition 6 in notes 14 **should** say that

"..*if* $x \in A$ and $0 < |x - c| < \delta(\varepsilon)$, then $|f(x) - L| < \varepsilon$."

(If your copy does not, change it by adding "$0 <$" before $|x - c|$. Do the same back in Definition 1.)

Give an example of a set A, a function f, a point c, and a number L, such that

$$\lim_{x \to c} f(x) = L$$

according to this definition, but this would not be true if the inequality $0 < |x - c|$ were omitted from the definition.

2. In definition 6 of notes 14, and Definition 4.1.4 in the text, it is required that c be a cluster point of A. Suppose we drop that requirement. Give an example which illustrates the fact that under the less restrictive definition, if c is not a cluster point of A, and f is any function from A to R, and L is any number, then $\lim_{x \to c} f(x) = L$. Hint: first you have to find a set A and a point c such that c is not a cluster point of A.

3. Suppose that we have two definitions, say definition A and definition B, of limit. Then mathematicians would say that definition A was "weaker" than definition B if every limit which existed according to definition B also existed according to definition A, but not vice-versa. (And definition B would be "stronger" than

definition A.) Would you say that the definition of limit in problem 2 was too strong to be useful, or too weak to be useful?

4. Use the definition of limit to prove that

$$\lim_{x\to 2} x^3 = 8.$$

5. Use the definition of limit to do problem 7, pg. 104.

6,7: pg. 104, # 9(b), 10(a) .

8. Define what is meant by

$$\lim_{x\to c^-} f(x) = -\infty.$$

Include any required properties of f and c. (For example, in Definition 1 in notes 14, I required that f be defined on some interval (a, b) and that $c \in (a, b)$. Definition 6 was less restrictive.) This type of limit is not explicitly defined in the text, though it is related to some definitions in section 4.3. Your definition should be reasonable in light of those definitions. Give an example illustrating your definition. Include a graph in your illustration.

9,10: pg. 110, # 2 (c,d)