Math 0450
Honors intro to analysis
Spring, 2009
Notes 14

1 Chapter 4, Limits

1.1 4.1 Limits of functions

Read first: 4.1 The simplest case is when (a,b) is an open interval in R, and $f:(a,b)\rightarrow R$. Suppose also that $c\in(a,b)$.

Definition 1 *We say that*

$$\lim_{x\rightarrow c} f(x) = L$$

if, for every $\varepsilon > 0$, *there is a* $\delta(\varepsilon) > 0$ *such that if* $0 < |x-c| < \delta(\varepsilon)$, *then* $|f(x)-L| < \varepsilon$.

Example 2 $(a,b) = (-\infty,\infty) = R$. *Show that*

$$\lim_{x\rightarrow 1}(2x-3) = -1.$$

Proof. Observe that for any $x\in R$,

$$|(2x-3)-(-1)| = |2x-2| = 2|x-1|.$$

Suppose that $\varepsilon > 0$. Let $\delta(\varepsilon) = \frac{\varepsilon}{2}$. If $|x-1| < \delta(\varepsilon)$, then

$$|(2x-3)-(-1)| = 2|x-1| < 2\frac{\varepsilon}{2} = \varepsilon.$$

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This example is very much like those we did for sequences. Unfortunately, there are times when finding $\delta(\varepsilon)$ can be somewhat harder than finding $K(\varepsilon)$ was for the examples we gave of sequences. Here is one such case:

Example 3 *Prove that*

$$\lim_{x\rightarrow 1} x^2 = 1.$$

This looks as simple as the previous example, but it is not. To see why, we start out as before:

$$\left|x^2-1\right| = \left|(x-1)(x+1)\right| = |x-1|\,|x+1|\,.$$

This presents a problem. Suppose that $\varepsilon > 0$. We want

$$|x-1|\,|x+1| < \varepsilon,$$

which is true for $x \neq -1$ if and only if

$$|x-1| < \frac{\varepsilon}{|x+1|}.$$

Because x appears on both sides of this inequality, it does not immediately tell us how small $|x-1|$ should be. The larger $|x+1|$ is, the smaller we need to choose $|x-1|$.

To get around this, we need to be sure that $|x+1|$ is not very large. We can accomplish this by making an initial assumption on $\delta(\varepsilon)$. This assumption is pretty arbitrary in this case. We will assume that no matter what ε is, we have $\delta(\varepsilon) < 1$. This is allowed because there is no necessity to choose the "best" $\delta(\varepsilon)$.

So assume that $|x-1| < 1$. Then

$$0 < x < 2,$$

and hence,

$$|x+1| < 3.$$

We can then say that

$$|x-1|\,|x+1| > 3\,|x-1|\,.$$

Now we prove our limit:

Proof. Suppose that $\varepsilon > 0$. Let $\delta(\varepsilon) = \min\left\{1, \frac{\varepsilon}{3}\right\}$. If $0 < |x-1| < \delta(\varepsilon)$, then in particular, $|x-1| < 1$, which implies that $|x+1| = |x-1+2| \le |x-1| + 2 < 3$. Hence,

$$\left|x^2-1\right| = |x-1|\,|x+1| < 3\,|x-1| < 3\frac{\varepsilon}{3} = \varepsilon.$$

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The proof is short, once we figure out the reasoning!

It seems unreasonable that this limit would be so hard, when for sequences we have a simple theorem: If $\lim(x_n) = 1$ then $\lim(x_n^2) = 1$. Fortunately, we can use reasoning with sequences to do the limits of functions also. The reason is the following theorem (basically, Theorem 4.1.8, pg. 101)

Theorem 4 *Suppose* $f : (a, b) \to R$ *and* $c \in (a, b)$. *Then*

$$\lim_{x \to c} f(x) = L$$

if and only if for every sequence (x_n) *such that (a) for every* $n \in \mathcal{N}$, $x_n \neq c$, *and (b)* $\lim (x_n) = c$, *it is the case that*

$$\lim (f(x_n)) = L.$$

I will apply this theorem below. But first, you will notice that there is some difference in wording between the theorem above and Theorem 4.1.8. This is because the text is discussing a more general definition of limit. In this definition, for example, the point c could equal a or b. The function f might not even be defined at c. Or, the function f might not be defined everywhere in an interval. Here is an example of such a function, which I will call "*den*" so that I can refer specifically to it later on.

If r is a rational number, $r \neq 0$, and $r = \frac{p}{q}$ in lowest terms, let

$$den(r) = \frac{1}{q}.$$

("*den*" refers to denominator) .This function is not defined at any irrational number. It is also not defined at $r = 0$. So what would be meant, for example, by

$$\lim_{r \to 0} f(r)?$$

Definition 5 *Suppose that* A *is a subset of* R. *Then a point* $c \in R$ *is called a "cluster point" of* A *if there is a sequence* $(x_n) \subseteq A$ *such that (a) for every* $n \in \mathcal{N}$, $x_n \neq c$, *and (b)* $\lim (x_n) = c$.

Notice that c need not be an element of A.

Definition 6 *Let* A *be a subset of* R, c *a cluster point of* A, *and* $f : A \to R$. *Then*

$$\lim_{x \to c} f(x) = L$$

if for every $\varepsilon > 0$ *there is a* $\delta(\varepsilon) > 0$ *such that if* $x \in A$ *and* $0 < |x - c| < \delta(\varepsilon)$, *then* $|f(x) - L| < \varepsilon$.

Remark 7 *The requirement ath $0 < |x - c|$ is important. If $0 = |x - c|$, then $x = c$, and $f(x)$ might not be defined. But even if it is defined, we don't want to require that $|f(c) - L| < \varepsilon$. The function could take a sudden jump when $x = c$, but the limit looks at nearby values of x, not $x = c$.*

Example 8 *Let*

$$\begin{aligned} &f(x) = x \text{ for all } x, \text{ and} \\ &g(x) = \begin{array}{l} x \text{ if } x \neq 0 \\ 1 \text{ if } x = 0. \end{array} \end{aligned}$$

Then

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) = 0.$$

Example 9 *Let $A = Q \setminus \{0\}$. Observe that 0 is a cluster point of A. Then $den : A \to R$. I claim that*

$$\lim_{r \to 0} den(r) = 0.$$

Proof: Suppose that $\varepsilon > 0$. Let $\delta(\varepsilon) = \varepsilon$. Suppose that $r \in Q$ and $0 < |r| < \varepsilon$. (Thus, $r \neq 0$.) Suppose that $r = \frac{p}{q}$ in lowest terms. We must have $|p| \geq 1$. I claim that $\left|\frac{1}{q}\right| < \varepsilon$. If not, then $|r| = \left|\frac{p}{q}\right| \geq \left|\frac{1}{q}\right| > \varepsilon$, a contradiction. Hence $|den(r) - 0| = \left|\frac{1}{q}\right| < \varepsilon$.

So now, we can use the text form of Theorem 4.1.8. We apply this to prove:

Theorem 10 *Suppose that A is a subset of R, $f : A \to r$ and $g : A \to R$. Suppose that c is a cluster point of A and that*

$$\begin{aligned} \lim_{x \to c} f(x) &= L_1 \\ \lim_{x \to c} g(x) &= L_2. \end{aligned}$$

Then

$$\lim_{x \to c} f(x) g(x) = L_1 L_2.$$

Remark 11 *Is the converse of this theorem true?*

Proof. Suppose that (x_n) is a sequence of points in A, with $x_n \neq c$ for all n. By Theorem 4.1.8,

$$\begin{aligned} \lim\left(f\left(x_n\right)\right) &= L_1 \\ \lim\left(g\left(x_n\right)\right) &= L_2. \end{aligned}$$

By theorem 3.2.3(a),

$$\lim\left(f\left(x_n\right) g\left(x_n\right)\right) = L_1 L_2.$$

Again applying Theorem 4.1.8,

$$\lim_{x \to c} f(x)\, g(x) = L_1 L_2.$$

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So what's neat is that we can prove all the same theorems about limits of functions that were proved in section 3.2 about limits of sequences.

And in particular, we can conclude that

$$\lim_{x \to 1} x^2 = 1. \tag{1}$$

Remark 12 *If, in homework or on the final, I ask you to prove a limit of a function "using the definition of limit", then you have to find* $\delta(\varepsilon)$. *(Otherwise you can use the theorems in 4.2.) You can rest assured that there will be such a problem on the final, but I may not feel that it is necessary to make it tricky, like* (1)*.*