

(Some revisions to the theory of the Cantor set were made on 2/22. Reprint if you have an earlier copy.)

1 Density in R

This topic is related to chapter two. It is a bit more sophisticated than the discussion in the book.

Recall the “density theorem”, #2.4.8, and its corollary.

Theorem 1 *If x and y are real numbers, with $x < y$, then there is a number $r \in Q$ such that $x < r < y$.*

Corollary 2 *If x and y are real numbers, with $x < y$, then there is a number $z \notin Q$ such that $x < z < y$.*

The term “density” comes from the following definition:

Definition 3 *A subset X of R is said to be a “dense” subset of R if for every $w \in R$, there is a sequence $\{x_n\}$ of numbers in X which converges to w .*

Example 4 *The set Q of rational numbers is a dense subset of R*

Proof. Suppose that $w \in R$. If w is rational, then just take the constant sequence (w, w, w, \dots) . If w is irrational, then for each n , choose a rational number x_n with $w - \frac{1}{n} < x_n < w + \frac{1}{n}$. This can be done because of the density theorem. Then by the squeeze theorem, (x_n) converges to w . ■

In the same way we can prove that the set of irrational numbers is dense in R . Notice that the set of rational numbers is countable. Hence, since R is uncountable, the set of irrational numbers must be uncountable. Thus, we have found both countable and uncountable dense subsets of R

We can extend the definition of density as follows:

Definition 5 *Let X be a subset of R , and Y a subset of X . Then Y is said to be “dense in X ” if for every $x \in X$, there is a sequence (y_n) of numbers in Y such that (y_n) converges to x .*

2 The Cantor set

2.0.1 read first: Section on Cantor set, 316-318

This is another example of the originality of the mathematician Georg Cantor. To construct the set named after him, start with the interval $[0, 1]$. From this interval, remove the subinterval $(\frac{1}{3}, \frac{2}{3})$. This leaves the set

$$\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

From this set remove $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$. This leaves

$$\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]. \quad (1)$$

Continue in this manner. For example, from $[0, \frac{1}{9}]$, remove $(\frac{1}{27}, \frac{2}{27})$. So at the end of the third stage, you have 8 closed intervals, and at the end of the n^{th} stage you have 2^n closed intervals. In this process, you are constructing many nested sequences of closed intervals, such as

$$[0, 1] \supset \left[\frac{2}{3}, 1\right] \supset \left[\frac{6}{9}, \frac{7}{9}\right] \supset \left[\frac{20}{27}, \frac{21}{27}\right] \supset \left[\frac{62}{81}, \frac{63}{81}\right] \supset \dots \quad (2)$$

The intersection $\bigcap_{n=1}^{\infty} I_n$ of each of these infinite nested sequences contains exactly one point. The set of points obtained in this way is called the “Cantor set”.

Here is another way to describe this set. Recall that we can write numbers in $[0, 1]$ as ordinary (base ten) decimals, such as $\frac{1}{2} = .499999 \dots$,¹ or in binary form, so that

$$\frac{1}{2} = .01111111 \dots = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

We can also write numbers in “ternary” form, meaning in base three. For example, $\frac{1}{3} = .022222 \dots$ in base three. If x is in $[0, 1]$ and has the ternary expansion

$$x = .\sigma_1\sigma_2\sigma_3 \dots$$

where $\sigma_n \in \{0, 1, 2\}$, then

$$x = \sum \left(\frac{\sigma_n}{3^n}\right).$$

The Cantor set can be described as the set of numbers which have no 1 in some form² of their ternary expansion.

¹We can write every decimal so that it has an infinite number of nonzero digits. Of course, $\frac{1}{2} = .5$ as well. In what follows we will need both types of expansions.

²See the previous footnote.

Theorem 6 *The Cantor set is uncountable.*

Suppose that \mathcal{F} is the Cantor set, and $f : \mathcal{N} \rightarrow \mathcal{F}$ is a bijection. Let $f(n) = c_n$. Each c_n has a ternary expansion, say

$$c_n = .\sigma_{n1}\sigma_{n2}\sigma_{n3}\cdots.$$

(We always choose the ternary expansion with no 1's, so that perhaps it ends in all zeros.) Define a number $c \in [0, 1]$ as follows:

$$c = \sum_{n=1}^{\infty} \frac{\sigma'_{nn}}{3^n}$$

where $c'_{nn} = \begin{cases} 0 & \text{if } c_{nn} = 2 \\ 2 & \text{if } c_{nn} = 0. \end{cases}$ Then $c \in \mathcal{F}$, since c has no 1's in its ternary expansion. But $c \neq c_n$ for any n . So f could not be a bijection.

Another way to see this is to define a bijection from \mathcal{F} to \mathcal{R} by letting

$$f\left(\sum \frac{\sigma_n}{3^n}\right) = \sum \frac{\delta_n}{2^n}$$

where

$$\delta_n = \begin{cases} 0 & \text{if } \sigma_n = 0 \\ 1 & \text{if } \sigma_n = 2 \end{cases}$$

It is obvious that \mathcal{F} is not dense in $[0, 1]$. For example, the whole interval $(\frac{1}{3}, \frac{2}{3})$ is missing. In fact, \mathcal{F} has what might be called the opposite property:

Definition 7 *Suppose that X is a subset of R . Then the "closure" of X is the set*

$$\bar{X} = X \cup \{\text{all limit points of sequences } (x_n) \subset X\}.$$

For example, the closure of the interval $(0, 1)$ is the interval $[0, 1]$. More interestingly, the closure of Q is R . (why?)

Definition 8 *A subset X of R is called "nowhere dense" if its closure contains no open intervals.*

Then the Cantor set is nowhere dense. To see this, suppose that $\bar{\mathcal{F}}$ contains an open interval (a, b) . If we choose n so that $\frac{1}{3^n} < b - a$, then one of the open intervals I which was thrown out in constructing \mathcal{F} must intersect the open interval (a, b) , and the center point of $I \cap (a, b)$ is not in $\bar{\mathcal{F}}$, which is a contradiction.

Thus, the Cantor set is uncountable and nowhere dense.

3 Homework (These problems are not easy. Start working on them as soon as you can. If you can't solve a problem, but think you can make some steps in the right direction, or have some useful comments, include those in what you hand in.)

1. Prove that $\mathcal{F} = \bar{\mathcal{F}}$. (In other words, \mathcal{F} is a “closed set”.) Hint: Suppose that either form of the infinite ternary expansion of a limit point of \mathcal{F} contains a 1.)

2. Suppose that r is a rational number. Let

$$A = \{nr + m : n, m \in \mathbb{Z}\}.$$

Prove that $A \cap [0, 1]$ is not dense in $[0, 1]$. Hint: For each $n \in \mathbb{Z}$, there is exactly one $m \in \mathbb{Z}$ such that $nr + m \in [0, 1)$. You can consider the case where $r > 0$.

3. Suppose that α is an irrational number. Let

$$B = \{n\alpha + m : n, m \in \mathbb{Z}\}.$$

Prove that $B \cap [0, 1]$ is dense in $[0, 1]$. Hint: For each $n \in \mathbb{Z}$ with $n \neq 0$, there is exactly one $m \in \mathbb{Z}$ such that $n\alpha + m \in (0, 1)$. Also, it is no loss of generality to assume that $\alpha > 0$. It might be helpful as well to consider the set you get if n is required to be positive, though you can't assume that in the final answer.

More on the Cantor set: In one way, the Cantor set is surprisingly small. Let's calculate how much of the interval $[0, 1]$ is not in the Cantor set. In other words, we add up the lengths of all of the subintervals which were removed. This will be an infinite series. We get

$$\begin{aligned} & \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \left(\frac{1}{3}\right) \left(\frac{1}{1-\frac{2}{3}}\right) = 1. \end{aligned}$$

Therefore, we say that the complement of the Cantor set, $[0, 1] \setminus \mathcal{F}$, has “measure” 1, while the Cantor set has measure 0. Thus, the Cantor set is an uncountable closed set with measure 0.

While we do not have time to give the general definition of “measure”, I want to mention that the set of irrationals in $[0, 1]$, while not closed, turns out to have measure 1, even though it has the same cardinality as \mathcal{F} . The rationals have measure 0. See page 311 of the text, where measure is presented in exercises 15 and 16.

Another property of the Cantor set is this:

Definition 9 A subset S of R is called a “perfect” set if it is closed ($\bar{S} = S$), and every point s in S is a limit point of some sequence $(s_n) \subset S \setminus \{s\}$.

Example 10 A closed interval, such as $[0, 1]$, is perfect. A finite set of points is not perfect.

Theorem 11 \mathcal{F} is perfect.

Proof. Homework problem 1 was to show that \mathcal{F} is closed. Now suppose $c \in \mathcal{F}$. Then

$$c = \sum \frac{\sigma_n}{3^n}$$

where each σ_n is either 0 or 2. Suppose that $\varepsilon > 0$. Pick $K \in \mathcal{N}$ so large that if $n \geq K$, then $\frac{1}{3^{n-1}} < \varepsilon$. Now change the next $((k+1)^{st})$ entry in the ternary expansion of c , from 0 to 2 or from 2 to 0. Leave the rest of the expansion the same. You obtain a number $s_n \in \mathcal{F}$ such that $s_n \neq c$ and $|s_n - c| < \varepsilon$. (This is because absolute value of the series for $s_n - c$ is less than $\sum_{n=K}^{\infty} \frac{2}{3^n} < \frac{1}{3^{K-1}} < \varepsilon$.) Hence $\{s_n\}$ is a sequence in \mathcal{F} which converges to c . ■

So \mathcal{F} is uncountable, of measure zero, nowhere dense, and perfect. Lots of sets are some of these, but essentially only something like the Cantor set is all of them.

Another such set is obtained if, instead of removing the middle $1/3$ of an interval each time, you remove some other fraction. If you remove $8/10$, then you get the set of all numbers whose infinite base 10 decimal expansion consists of only 0's and 9's.