Math 0450
Honors intro to analysis
Spring, 2009

# 1 More concise proof of part (a) of the monotone convergence theorem.

**Theorem 1** *If* $(x_n)$ *is a monotone and bounded sequence, then* $\lim(x_n)$ *exists.*

**Proof.** (a) Suppose that $(x_n)$ is increasing. Let $S = \{x_n : n \in \mathcal{N}\}$ and let $x = \sup S$, which exists because $S$ is a bounded set. I claim that $\lim(x_n) = x$. To prove this, suppose that $\varepsilon > 0$. Then $x - \varepsilon$ is not an upper bound for $S$, by the definition of supremum. Hence there is an $n_1 \in \mathcal{N}$ such that $x - \varepsilon < x_{n_1}$. But $x$ is an upper bound for $S$, so $x - \varepsilon < x_{n_1} \leq x$. Hence $|x_{n_1} - x| < \varepsilon$. Let $K(\varepsilon) = n_1$. If $n \geq K(\varepsilon)$, then $x_{n_1} \leq x_n \leq x$, since we assumed that $(x_n)$ is increasing. Hence $|x_n - x| \leq |x_{n_1} - x| < \varepsilon$. This proves that $\lim(x_n) = x$. ■

Remark: I haven't stated specifically that the $x_n$ are real numbers; that is understood. Sometimes I wrote "$n \in \mathcal{N}$", or $n_1 \in N$ because it was easy to fit in and didn't disrupt the flow. But I avoided the phrase "$n$ is a natural number", because that is awkward. Usually we understand from the context that $x$ is a real number, $n$ a positive integer. If we want to make it clear that negative integers are also included, we could write $n \in Z$. If we want only non-negative integers, we could write $n \geq 0$. Generally, again depending on context, $x \geq 0$ would be understood to mean real numbers, not just integers.

# 2 Infinite series

Definition: Suppose that $(x_n)$ is a sequence of real numbers. Then $\sum x_n$ denotes another sequence, called the "sequence of partial sums" of $(x_n)$, and defined by

$$\begin{aligned} s_1 &= x_1 \\ s_{n+1} &= s_n + x_{n+1} \text{ for } n \geq 1. \end{aligned}$$

This sequence is also called an "infinite series", and may be denoted as well by $\sum_{n=1}^{\infty} x_n$. The series is said to "converge" if the sequence $(s_n)$ converges, according to the usual definition of convergence of a sequence.

Often, it is assumed that the sequence $(x_n)$ starts with $n = 0$, and to make this clear, the series is written as $\sum\limits_{i=0}^{\infty} x_i$.

Example: $(x_n) = \frac{1}{2^n}$, for $n \geq 0$. Then

$$s_n = \sum_{i=0}^{n} x_i = \sum_{i=0}^{n} \frac{1}{2^i} = 1 + \frac{1}{2} + \frac{1}{4} + \cdot \cdot \cdot + \frac{1}{2^n}. \tag{1}$$

Hence,

$$s_1 = 1,\ s_2 = \frac{3}{2},\ s_3 = \frac{7}{4}.$$

By a formula from chapter 2, the $n^{\text{th}}$ partial sum is

$$s_n = \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}}.$$

From this it is easy to see that $\lim (s_n) = 2$. Often this fact is written as

$$\sum_{i=0}^{\infty} \frac{1}{2^i} = 2.$$

Again, this must be understood as saying that the sequence of partial sums, $(s_n)$, converges to 2.

Example: (pg. 90, Example 3.7.2(c) ) Consider the series $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)}$. Discuss its convergence or divergence.

I will do this more formally than the text, by using the definition of the sequence of partial sums. For this example,

$$\begin{aligned} s_1 &= \frac{1}{2} \\ s_2 &= \frac{1}{2} + \frac{1}{6} \\ s_3 &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} \end{aligned}$$

and

$$s_{n+1} = s_n + \frac{1}{(n+1)(n+2)}.$$

The key is the partial fractions formula

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \tag{2}$$

Thus, for example,

$$s_3 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = 1 - \frac{1}{4}.$$

(This is a “collapsing sum”, or a “telescoping sum”.) From this we have a lemma:

**Lemma 2** $s_n = 1 - \frac{1}{n+1}$.

**Proof.** Since $s_1 = 1 - \frac{1}{2}$, the lemma is true for $n = 1$. Suppose it is true for $n = k$. Then,

$$s_{k+1} = s_k + \frac{1}{(k+1)(k+2)} = s_k + \left(\frac{1}{k+1} - \frac{1}{k+2}\right).$$

From the induction hypothesis,

$$s_{k+1} = \left(1 - \frac{1}{k+1}\right) + \left(\frac{1}{k+1} - \frac{1}{k+2}\right) = 1 - \frac{1}{k+2}.$$

This proves the lemma. Now the result is obvious:

$$\lim(s_n) = 1.$$

■

## 2.1 The harmonic series.

This is the series

$$\sum \frac{1}{n}.$$

It is a very important result that this series diverges. More precisely, if $(s_n)$ is the sequence of partial sums, then

$$\lim(s_n) = \infty$$

(See section 3.6.)

**Proof**: We consider the subsequence

$$\left(s_{(2^n)}\right) = \left(1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}, 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}, ..\right)$$

It helps to see what is going on if we group the terms appropriately, so that the last term is

$$s_3 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8}.$$

**Lemma**: $s_{2^n} \geq \frac{n+1}{2}$.

**Proof**: $s_2 = \frac{3}{2} = \frac{2+1}{2}$. Now suppose the formula is true for $n = k$. Then I claim that

$$s_{2^{k+1}} = s_{2^k} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} \geq \frac{k+1}{2} + 2^k \frac{1}{2^{k+1}} = \frac{k+2}{2}.$$

This is because $\left(2^k + 2^k\right) = 2^{k+1}$, so that $s_{2^{k+1}}$ has $2^k$ more terms than $s_{2^k}$, and each of those terms is greater than or equal to the last term, which is $\frac{1}{2^{k+1}}$. This proves the lemma.

The divergence of the series follows because a subsequence is unbounded. Since the sequence of partial sums is monotone increasing, this proves that $\lim (s_n) = \infty$ .

# 3 About the exam

The exam will cover through page 90. A problem like the examples on page 90, or like problems 8 and 9 below may be chosen for the exam. Also, problems 3(a), (4) and (5) would be possibilities. The harmonic series is example 3.5.6(c), pg. 84, and so fair game, though I do it differently above.

Regarding earlier material, if you are asked to prove something using the properties of real numbers from chapter 2, you will be given a list of those properties.

If, in chapter 3, you are asked to prove that a sequence converges or diverges "using the definition of convergence," then you must find $K(\varepsilon)$ or show that no $K(\varepsilon)$ exists. You cannot use the theorems in section 3.2 or later.

# 4 Homework

**This assignment is due on Tuesday, Feb. 17, in advance of the exam on Feb. 18.**

Page 80: # 7(b), 8(b), 12

pg. 86, # 5, 8, 13

pg. 88, # 2

8. Prove that the sequence $(s_n)$ of partial sums for the sequence $(x_n) = \left(\frac{1}{n!}\right)$ is a Cauchy sequence, using the definition of Cauchy sequence. Hint: Use problem 14 on page 16.

9. Determine if $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converges.