.Math 0450
Honors intro to analysis
Spring, 2009
Notes #11

1 The number e

On page 73 the text introduces the important number "e", or "Euler's number". The definition is based on the following result:

Proposition 1 *The sequence* $\left(\left(1+\frac{1}{n}\right)^n\right)$ *converges to a limit* L, *with* $2 < L < 3$.

Proof. I don't have a different proof from the text, but this is a good place to start using summation notation, that is, $\sum$, instead of "$+\cdots$" . It's not that $\sum$ is always better, but it seems to me to lend an air of authority to a proof!

If (x_j) is a sequence, then a new sequence, denoted by $\sum_{j=1}^{n} x_j$, can be defined inductively:

$$\sum_{j=1}^{1} x_j = x_1$$
$$\sum_{j=1}^{n+1} x_j = \sum_{j=1}^{n} x_j + x_{n+1}$$

Eventually (in section 3.7), we will discuss when the sequence $\sum_{j=1}^{n} x_j$ converges, but for now, we use the notation to prove the proposition.

Presumably you are familiar with this, but you may be less familiar with a corresponding notation for products. Again, if (x_j) is a sequence, define $\prod_{j=1}^{n} x_j$ by

$$\prod_{j=1}^{1} x_j = x_1$$
$$\prod_{j=1}^{n+1} x_j = x_{n+1} \prod_{j=1}^{n} x_j .$$

(I'm probably being overly pedantic here. When it comes down to it,

$$\sum_{j=1}^{n} x_j = x_1 + \cdots + x_n, \quad \prod_{j=1}^{n} x_j = x_1 x_2 \cdots x_n.$$

But just so you see we really don't have to use $\cdots$, I went through the formal definition. Really, $\cdots$ is just a shorthand for induction.)

When using $\sum$ and $\prod$, we often want to start with $j = 0$ instead of $j = 1$. Similarly, many sequences start with x_0. We do this without comment, realizing that some shift of index is necessary if we really want to insist that a sequence is a function from $\mathcal{N}$ to R.

We also use the binomial coefficients, $\left(\begin{array}{c} n \\ j \end{array} \right) = \frac{n!}{j!(n-j)!}$. These can also be written using product notation:

$$\left(\begin{array}{c} n \\ j \end{array} \right) = \frac{\prod_{i=0}^{j-1} (n-i)}{j!}.$$

With these coefficients, the binomial theorem becomes

$$(a+b)^n = \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) a^{n-j} b^j$$

Now we can prove the proposition. We first prove that the sequence $(e_n) = \left((1 + \frac{1}{n})^n \right)$ is increasing. From the binomial theorem, we have

$$e_n = \left(1 + \frac{1}{n}\right)^n = \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) \frac{1}{n^j} = \sum_{j=0}^{n} \frac{1}{j!} \frac{\prod_{i=0}^{j-1} (n-i)}{n^j} = \sum_{j=0}^{n} \left(\frac{1}{j!} \prod_{i=0}^{j-1} \left(1 - \frac{i}{n}\right) \right).$$

In the last step, we divided each of the j terms in the product $\prod_{i=0}^{j-1} (n-i)$ by n.

Also,

$$e_{n+1} = \sum_{j=0}^{n+1} \left(\frac{1}{j!} \prod_{i=0}^{j-1} \left(1 - \frac{i}{n+1}\right) \right)$$

All the terms in the sum are positive. There is one more term in e_{n+1} than in e_n, and if $0 \leq i \leq n$, then

$$\left(1 - \frac{i}{n+1}\right) \geq \left(1 - \frac{i}{n}\right).$$

Hence, $e_{n+1} > e_n$.

Next we prove that $(e_n) = \left(\left(1+\frac{1}{n}\right)^n\right)$ is bounded above. Again using the binomial expansion, we get

$$e_n = \sum_{j=0}^{n} \left(\frac{1}{j!} \prod_{i=0}^{j-1} \left(1 - \frac{i}{n}\right)\right) \leq \sum_{j=0}^{n} \left(\frac{1}{j!}\right), \tag{1}$$

since each term in the product is less than or equal to 1. Hence, for $n \geq 2$,

$$e_n \leq 1 + 1 + \sum_{j=2}^{n} \left(\frac{1}{j!}\right) \leq 2 + \sum_{j=1}^{n-1} \left(\frac{1}{2^j}\right).$$

(Look at the first few terms of the sum to see this.)

On page 14 it is shown that if $r \neq 1$, then

$$\sum_{j=0}^{n} r^j = \frac{1 - r^{n+1}}{1 - r},$$

and using this with $r = \frac{1}{2}$ gives

$$e^n \leq 2 + \left(\frac{1 - \left(\frac{1}{2}\right)^n}{\frac{1}{2}} - 1 \right) < 3.$$

Hence, $\left(\left(1+\frac{1}{n}\right)^n\right)$ converges, and we denote the limit by e.

From what we have shown, we could have $e = 3$ (why?), or for that matter, $e = 1$. But it is not much more work to get that $2 < e < 3$. It is quite a bit more work to show that e is an irrational number, and still more to show that it is transcendental.

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