

## 1 Remarks on a limit

Suppose that you were asked to determine  $\lim \left( n^{\frac{1}{n}} \right)$ . One way to get started is to evaluate this function using a calculator. You can easily find that if  $f(n) = n^{\frac{1}{n}}$ , then

$$\begin{aligned} f(1) &= 1 \\ f(2) &= \sqrt{2} = 1.414\dots \\ f(3) &= 1.442\dots \\ f(4) &= 1.414\dots \\ f(5) &= 1.38\dots \end{aligned}$$

This is rather peculiar behavior: The function first goes up and then goes down! At this point you might get impatient, and jump to

$$\begin{aligned} f(10) &= 1.25 \\ f(100) &= 1.047 \end{aligned}$$

Since it's obvious that  $n^{\frac{1}{n}} > 1$ , you might now suspect that the limit is 1. Your goal now is to prove this.

Having had calculus, you might think about logarithms, since they help simplify a function with exponents. So you find:

$$\ln \left( n^{\frac{1}{n}} \right) = \frac{\ln n}{n}.$$

Then you might recall that  $\ln x$  is a slower growing function than  $x$ , and guess that  $\lim \frac{\ln n}{n} = 0$ . Hence, you guess,

$$\lim \left( n^{\frac{1}{n}} \right) = e^{\lim(\ln(n^{1/n}))} = e^0 = 1.$$

It's consistent. To prove that  $\lim \left( \frac{\ln n}{n} \right) = 0$  you could try L'hospital's rule, thinking of  $n$  as a continuous variable, like  $x$ .

Eventually we will see that this is all correct. But right now, we have not had logarithms, and certainly can't claim to have proved L'hospital's rule. So a proof must be based on something else. Here is where a clever idea is needed!

The text tells us the clever idea: Write  $n^{\frac{1}{n}} = 1 + k_n$ , and try to show that  $\lim(k_n) = 0$ . As we saw in class, it is tempting to say:

$$n = \left(n^{\frac{1}{n}}\right)^n = (1 + k_n)^n \quad (1)$$

and then use Bernoulli's inequality. We get  $n \geq 1 + nk_n$ . We want to show that  $k_n \rightarrow 0$ , so we rewrite this as

$$k_n \leq \frac{n-1}{n}. \quad (2)$$

But then we see, as I saw in class, that this is not good enough. The power of  $n$  in the denominator is the same as the power of  $n$  in the numerator, so the limit on the right is not zero. We have done several like this, so we can probably see that  $\lim\left(\frac{n-1}{n}\right) = 1$ , which does not help with showing that  $\lim(k_n) = 0$ .

So we have to do better. Instead of equation (2) above, we need something with  $n$  in the numerator and  $n^2$  in the denominator. Looking at (1), the binomial theorem may come to mind, to get some more information about  $(1 + k_n)^n$ . Recall the binomial theorem:

$$(a + b)^n = a^n + \frac{n!}{(n-1)!1!}a^{n-1}b + \frac{n!}{(n-1)!2!}a^{n-2}b^2 + \dots + b^n.$$

Simplifying those factorials, we get

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \dots + b^n.$$

So we get

$$n = (1 + k_n)^n = 1 + nk_n + \frac{n(n-1)}{2}k_n^2 + \dots + k_n^n. \quad (3)$$

This is too complicated, but we don't need to figure out exactly what  $k_n$  is. We only want to show it is less than something. So we need an inequality which gives us  $k_n \leq$  something. Showing that  $k_n^2 \rightarrow 0$  is good enough, by one of the limit theorems involving square root, and the  $k_n^2$  term above also has an  $n^2$ , which is what we were looking for. So we simplify the right side of (3) by dropping all of the terms except the third term. When we do that, we make the right side smaller, and so we get

$$n \geq \frac{n^2 - n}{2}k_n^2.$$

Now we have

$$k_n^2 \leq \frac{2n}{n^2 - n}$$

if  $n > 1$ . Now we have a bigger power of  $n$  in the denominator. Since  $k_n^2 \geq 0$ , we can write

$$0 \leq k_n^2 \leq \frac{2n}{n^2 - n} = \frac{2}{n - 1}$$

if  $n > 1$ .

I leave the rest of the proof to you.

### 1.0.1 Read 3.2

In 3.2 there are general theorems, most of which are pretty obvious, which can be used in giving rigorous proofs of limits. Here is an example of such a problem:

pg. 67, # 1(d). Find  $\lim \left( \frac{2n^2+3}{n^2+1} \right)$ . First divide top and bottom by  $n^2$ . This is just algebra, and having finished chapter 2, we don't need to cite any theorems in justification:

$$\lim \left( \frac{2n^2 + 3}{n^2 + 1} \right) = \lim \left( \frac{2 + \frac{3}{n^2}}{1 + \frac{1}{n^2}} \right).$$

The remaining steps are:

By 3.1.6(a),  $\lim \left( \frac{1}{n} \right) = 0$ , so by 3.2.3(a),  $\lim \left( \frac{3}{n} \right) = 0$ . Since  $0 \leq \frac{3}{n^2} \leq \frac{3}{n}$ , the squeeze theorem (3.2.7) implies that  $\lim \left( \frac{3}{n^2} \right) = 0$ . Also,  $\lim (2) = 2$ , (As the book says in one place, this is "evident". It is a simple consequence of the definition of limit, which we will not comment on any more!) By 3.2.3(a),  $\lim \left( 2 + \frac{3}{n^2} \right) = 2$ . Similarly,  $\lim \left( 1 + \frac{1}{n^2} \right) = 1$ , and since  $1 \neq 0$  (a theorem in chapter 1!), 3.2.3(b) implies that  $\lim \left( \frac{2n^2+3}{1+\frac{1}{n^2}} \right) = 2$ .

As usual, once we have finished this section, such detailed citations will not be necessary.

One Theorem in section 3.2 is surprisingly hard to prove:

**Theorem 3.2.10.** *If  $(x_n)$  is a sequence which converges to  $x$ , and  $x_n \geq 0$  for every  $n$ , then  $\lim (\sqrt{x_n}) = \sqrt{x}$ .*

I will go over the proof in class. Here I want to note an algebraic identity which will be useful in the homework as well as this proof:

$$a - b = \frac{a^2 - b^2}{a + b}.$$

Here is another important principle:

If  $\lim (x_n) = x$ , then for any  $k \in \mathbb{N}$ ,  $\lim (x_{n+k}) = x$ . (See the section on “tails” of sequences on pg. 57)

This follows from the definition of convergence. If  $K(\varepsilon)$  works for  $(x_n)$ , and  $n \geq K(\varepsilon)$ , then  $n + k \geq K(\varepsilon)$ , so the same  $K(\varepsilon)$  shows that  $(x_{n+k})$  converges.

### 1.0.2 Read 3.3.

One interesting thing in this section is the introduction of the number  $e$ , (named for Euler). Often this is only encountered after defining logarithms in terms of integrals. But here it is defined to be a limit.

**Example 3.3.6:** The sequence  $(1 + \frac{1}{n})^n$  converges.

Notation:  $\lim (1 + \frac{1}{n})^n$  is denoted by “ $e$ ”.

The notation needs no proof. This is simply the definition of the number  $e$ . What needs proof is that the sequence converges. This is proved by using the monotone convergence theorem, Theorem 3.3.2. (Stated and proved in class). The proof that the sequence is monotone is tricky the proof that it is bounded above is easier. Both are in the text, and will be done in class if time permits.

Having defined  $e$ , it would be nice to know something about it. All that one can conclude from the convergence proof in the text is that  $2 \leq e \leq 3$ . It might be that  $e = 3$ . (I have read that the “ancients” thought that  $\pi = 3$ . This was well before  $e$  had been thought of, however. ) Clearly, if  $e = 3$  then there is no need for the notation  $e$ . It is, of course, possible to prove that  $2 < e < 3$ . We will do so eventually. As we discussed in class on the first day,  $e$  is a “transcendental” number. This means more than just “irrational”, though it is irrational. “Transcendental” means that  $e$  does not satisfy any polynomial equation  $p(x) = 0$  where

$$p(x) = \sum_{n=0}^{\infty} a_n x^n$$

and the  $a_n$  are rational numbers.

### Homework (Due Feb. 12)

pg. 67, # 6(d). (Prove, citing previous results in the text to justify each step)

pg. 67 #9, 10(a), 13(b), 15(d), (18(d)

For all these, cite reasons if they are not reasons you used in # 6(d). So you don't have to keep citing Theorem 3.2.3.) But if you are dealing with a fraction, you must verify explicitly that the denominator does not converge to 0. In this chapter you should also cite the squeeze theorem each time you use it.

pg. 74, # 4, 11, 13(d) .