.Math 0450
Honors intro to analysis
Spring, 2008
Notes #1
Corrected; The only change is to change a 6 to a 5 in the middle of page 2. The previous "f" was not injective.

These notes are supplementary to the text, and do not replace it. I will post them on the website, at irregular intervals. Not all material in the text, or in the lectures, will be discussed in the notes. Ordinarily, you should read the text first, then the relevant notes, if any. This first set is concerned with the material in chapter one of the text.

1 Sets and Functions

Read first: Section 1.1

1.1 Sets

Mathematics is a logical system of thought. As such, it has to start with some undefined terms and some axioms about these terms. Everything that follows has to be proved by reasoning from these axioms, or from previous theorems which were based on the axioms.

In this course the undefined terms are: "set", "natural number", "integer","rational number", and "real number". There are other courses, often taught be specialists in mathematical logic, which derive everything from the axioms for a set, but in order to get anywhere in analysis, we have to take numbers for granted. In chapter two the axioms for real numbers are discussed at length. The axioms for sets are not discussed, but on page 1 some basic notation about sets is introduced. We will use this notation frequently.

On pages 2-3 there are several definitions (one of which has three parts), followed by a theorem. You should read the proof of the theorem carefully, and see how it follows from the definitions. This process is followed throughout the book, and you will learn how to use the same process in homework.

As an illustration, I will do problem 1 on page 11:

Proposition 1 *If A and B are sets, show that $A \subseteq B$ if and only if $A \cap B = A$.*

Proof. (i) Suppose that $A \subseteq B$. We wish to show that $A \cap B = A$. From the remark below Definition 1.1.1 we see that we must show that $A \cap B \subseteq A$ and $A \subseteq A \cap B$.

First suppose that $x \in A \cap B$. Then by Definition 1.1.3(b), $x \in A$. Since x was an arbitrary element of $A \cap B$, this shows that $A \cap B \subseteq A$.

Now suppose that $x \in A$. By the hypotheses of step (i), $x \in B$. Hence $x \in A \cap B$. Since x was an arbitrary element of A, this shows that $A \subseteq A \cap B$, and so $A \cap B = A$, as desired in step (i) .

(ii) Suppose that $A \cap B = A$. We must show that $A \subseteq B$. Suppose that $x \in A$. Then by the hypothesis of step (ii), $x \in A \cap B$, and so $x \in B$. Hence, $A \subseteq B$. This completes the proof of step (ii) and of the proposition. ■

Several of your homework problems due next week are of this type. Be sure to justify every statement by reference to a definition or theorem in the text, or else something you proved previously in the assignment. (Later you can also refer to previous assignments.)

1.2 Functions

Recall from the text the definition of a function. A function f from a set A to a set B is a subset of the set of all ordered pairs (a, b) with $a \in A, b \in B$. That is, it is a subset of the "Cartesian product" $A \times B$. To be a function, this subset must have the property that no $a \in A$ appears in more than one of the ordered pairs of the function. If $(a, b) \in f$, then we write $b = f(a)$.

Another way to look at this is that the function is what in calculus we call the "graph of the function". The graph is a set of points in the x, y plane, or in other words, a subset of $A \times B$. To be a function, no vertical line can intersect the graph more than once.

The text describes different properties of functions, such as "injective", "surjective", and "bijective". Sometimes we call these "one to one", "onto", and "one to one and onto", respectively.

Definition 2 *A function f with domain A is called "injective" if, for each x_1 and x_2 in A with $x_1 \neq x_2$, it is the case that $f(x_1) \neq f(x_2)$.*

For the usual functions studied in calculus, which have graphs in the x, y plane, this means that no horizontal line can intersect the graph of f more than once. But in this class, we discuss other sorts of functions as well. In particular, we discuss functions whose domain is "discrete", such as $\mathcal{N}$, or a finite set such as $\{1, 2, 3\}$.

Here are two examples. One is injective (1:1), one is not:

$$A = \{1, 2, 3\}, B = \{4, 5, 6\}$$
$$f = \{(1, 5), (2, 4), (3, 6)\}$$

$$A = \mathcal{R}. \ B = \mathcal{R}$$
$$g = \left\{(x, y) \ : \ x \in \mathcal{R}, \ y = x^2\right\}.$$

Notice that $(1, 1)$ and $(-1, 1)$ are both in g, which is why g is not injective.

Recall also that $f : A \to B$ is surjective (onto) if $f(A) = B$. Even if f is not surjective, we can get a surjective map by setting $\hat{B} = f(A)$, and defining $\hat{f} : A \to \hat{B}$ by $\hat{f}(x) = f(x)$ for all $x \in A$.

Finally, recall that f is a bijection if it is both injective and surjective (1:1 and onto).

Homework: due at the beginning of class on January 14.

pg. 11, # 2. (Read the proof of Theorem 1.1.4 (a) on pg. 3. Be sure you understand what the authors mean by "conversely" in their proof.)

pg. 11, # 6, 8. (These problems do not say "prove", or "show". However in #8, and in most problems, you should give some reason for your answers, such as a graph.)

pg. 11, # 14,16,20.

Remark: I expect to assign between 5 and 10 problems each week. If there are more than 5, some will be short, and easy if you think about them correctly. In this assignment, #16 is short and easy.

2 Induction

Read first: section 1.2 By "induction" I always mean what the text calls the Principle of Mathematical Induction, which is actually a theorem, 1.2.2 on page 12. A second version of this theorem, which is often what we use in practice, is given at the top of page 13.

The proof of 1.2.2 depends on an axiom satisfied by the natural numbers $\mathcal{N}$, the "Well-ordering property", 1.2.1.

Several examples of the use of induction are given in the text. I will give another example, which I will word in three different ways, so you can see the relation between the statement of 1.2.2 and the version given at the top of page 13.

The example looks ahead a bit, to calculus. But you should be able to follow it.:

Proposition 3 *For* $n \in \mathcal{N}$,

$$\frac{d}{dx}x^n = nx^{n-1} \tag{1}$$

.

Remark 4 *We will assume that this has been proved for* $n = 1$*:* $\frac{d}{dx}x^1 = 1$*. This is very easy to prove from the definition of derivative, on pg. 158. We will also assume the product rule for derivatives, proved on page 160.*

We give three equivalent proofs of the proposition. The first uses 1.2.2. The second uses the reformulation at the top of page 13. The third is a rewording of the second in language closer to what a mathematician would ordinarily use.

Proof 1. Let $S = \{n\ :\ n \in \mathcal{N},\ \text{and}\ (1)\ \text{is true}\}$. We are assuming that $1 \in S$. Now assume for some $k \in \mathcal{N}$ that $k \in S$. Then

$$\frac{d}{dx}x^{k+1} = \frac{d}{dx}xx^k = 1x^k + x\frac{d}{dx}x^k = x^k + xkx^{k-1}, \tag{2}$$

since $k \in S$. But then,

$$\frac{d}{dx}x^{k+1} = x^k + kx^k = (k+1)\,x^k, \tag{3}$$

so $(k+1) \in S$. By 1.2.2, $S = \mathcal{N}$, so (1) holds for every $n \in \mathcal{N}$. ■

Proof 2. Let $P(n)$ be the statement that $\frac{d}{dx}x^n = nx^{n-1}$. We are assuming that $P(1)$ is true. Suppose that $P(k)$ is true. Then the calculations (2) and (3) show that $P(k+1)$ is true. Hence, by the version of the Principle of Mathematical induction at the top of page 13, $P(n)$ is true for every n. ■

Proof 3. We are assuming that (1) is true for $n = 1$. Suppose it is true for $n = k$. Then the calculations (2) and (3) show that it is true for $n = k + 1$. Hence, by induction, it is true for all positive integers n. ■

Remark 5 *If you are confused by the 2nd or 3rd proofs, use the method of Proof 1. I think that this is the most concrete and so easiest to understand.*

3 Infinity

Read first: section 1.3. One of the important ideas that you will learn to understand in this course is that of "infinity". Mathematicians believe that they are the only ones who understand this concept clearly! To start with, we ask:

What is an infinite set?

Before proceeding, you may wish to try to answer this for yourself. Assuming you have read the text, do you recall the definition of infinite set given there? Can you think of an alternative (but equivalent) definition? (Two definitions of "infinite set" are equivalent if every set which is infinite according to one definition is also infinite according to the other one.)

Here is one definition of an "infinite set" :

Definition 6 *A set A is infinite if there is an injective function $f : A \to A$ such that $f(A)$ is a proper subset of A. That is, $f(A) \subseteq A$ but $f(A) \neq A$.* [1]

This definition is different from the one given in the text. We should ask if the two definitions are equivalent. To do this, recall the definition of infinite set in the text: A set A is infinite if it is not finite. Obviously, we need to know what a finite set is: In the text, a set is finite if there is an bijection $f : A \to N_n = \{1, 2, 3, ..., n\}$, for some positive integer n.[2]

We have the following theorem:

Theorem 7 *A set A is infinite (according to our definition above) if and only if there is no bijection from A to N_n, for any positive integer n.*

To understand the proof of this theorem, you must be familiar with the definitions on pages 8 and 9 of the text. If you don't understand a term used below, stop and read about it on those pages. We also use the "uniqueness theorem", theorem 1.3.2 on page 17. This theorem is equivalent to saying that there is no bijection from N_n to N_m, if $m \neq n$. **The proof of this, given in Appendix B, uses the method of mathematical induction given in section 1.2.**

Proof. Suppose that A is infinite. Then there is a bijection $f{:}A \to B$, where $B \subseteq A$ and $B \neq A$. ($B = f(A)$, because f is a bijection.) But suppose also that for some n, there is a bijection g from A to N_n. I claim that $g \circ f \circ g^{-1}$ is a bijection from N_n to some proper subset C of N_n. To see this, observe that $g^{-1}(N_n) = A$, $f \circ g^{-1}(N_n) = B \neq A$, and so $g \circ f \circ g^{-1}(N_n) = g(B) \neq g(A) = N_n$. The last inequality is because g is a bijection and $B \neq A$.

The set C contains natural numbers $j_1, ..., j_m$ with $m < n$. We then define a further function h, by setting $h(j_k) = k$, and this gives a bijection $h \circ g \circ f \circ g^{-1}$ from N_n to N_m. But by theorem B.1 on page 343 (Appendix B), this is impossible. This contradiction proves Theorem 7. ∎

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[1] Note that $f : A \to A$ means that f is a function with domain A, and $f(A) \subseteq A$.

[2] As pointed out in the text, the positive integers are also not defined here. In fact, "real numbers" are not defined, though we will make a list of their properties. The fact that there is some set with the properties we will list, and that also such sets are equivalent in a certain sense, is left once again to a course on set theory.