Math 0290, Differential Equations  
Class Notes

1. What is a derivative? 
Definition: 
\[ f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \]

or 
\[ f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \]

We will more often use the notation \( x(t) \) for the function. "t" is the “independent variable” and \( x \) is the dependent variable.

\[ x'(t_0) = \lim_{h \to 0} \frac{x(t_0 + h) - x(t_0)}{h} \]

Other possible notations are \( u(t), v(s), y(x) \), etc.

\[ u'(t) = \lim_{s \to 0} \frac{u(t + s) - u(t)}{s} \]

\[ \frac{dv}{ds} = \lim_{t \to s} \frac{v(t) - v(s)}{t - s} \]

Interpretations:
(i) \( f'(x_0) \) is the slope of the tangent line to the graph of 
\[ y = f(x) \]

at the point \((x_0, f(x_0))\)
(ii) $x'(t)$ is the "rate of change" of the function $x$ at the point $t$.

$$x'(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}.$$ 

In this formula $\Delta t$ is thought of as a "small change in $t$". It is the same as "$h". Not zero, but small.

If $x$ is distance and $t$ is time, then the fraction

$$\frac{x(t + \Delta t) - x(t)}{\Delta t}$$

is the change in distance divided by the change in time, which gives the average speed.

As $\Delta t$ tends to zero, this fraction tends to the "instantaneous rate of change" of $x$ with respect to $t$, or the "velocity".

$$v(t) = x'(t) = \frac{dx}{dt}.$$ 

We also have the acceleration:

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2} = x''(t).$$

2. What is a differential equation?

Examples of equations in mathematics:

$$3(x + 1) = 3x + 3$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$e^{x+y} = e^x e^y$$

These are "identities". They are true for every value of the variables which appear. We don’t try to solve them for $x$ or $y$ or $\theta$.

$$x^2 + 5x - 2 = 0$$

This is an equation which is not true for every $x$. We hope that we can “solve for $x$”. This means to find every possible number $x$ for which this equation is true. In this case there are two solutions:

$$x_1 = \frac{-5 + \sqrt{25 + 8}}{2}$$

$$x_2 = \frac{-5 - \sqrt{25 + 8}}{2}.$$
Sometimes there may be no solutions:
\[ e^x = 0 \]
or infinitely many solutions:
\[ \sin x = 1 \]

Two more equations:
\[ 3x + 2y = 4 \]
\[ ax^2 + bx + c = 0 \]

Here there are several unknown quantities \((x, y, a, b, c)\). We can try to solve for one in terms of the others:
\[ y = \frac{4 - 3x}{2} \]
\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\(a, b, c\) are called “parameters” in the formula for \(x\).

Here is an equation with some derivatives
\[ \frac{u'}{v} - \frac{v'u}{v^2} = \left( \frac{u}{v} \right)' \]

This is also an identity. It is true for any function \(u\) and \(v\).

A “differential equation” is an equation where there is an unknown, but the unknown is a function, not a single number or set of numbers. The equation involves the derivatives of this unknown function.

\[ u' (x) = 3x \]

We would like to find a formula for the function \(u(x)\). And in this example this is easy.

\[ x' (t) = x (t) \]

Here we want to find a formula for the function \(x(t)\).
In this equation we are not trying to solve for “t”. This letter is not important, and can be changed without changing the equation. It is often omitted.

\[ x' = x \]

In words this equation says: The rate of change of \( x(t) \) at any time \( t \) is equal to the value of \( x(t) \).

Another example: The rate of change of \( y \) with respect to \( x \) is equal to \( y^2 \). The ode is

\[ y' = y^2. \]

Here is a “second order equation”:

\[ u'' + u' + u = 0 \]

Find a formula for the function \( u \).

(\( u(x) \), or \( u(t) \) or \( u(s) \) – it doesn’t matter.)

All of these are “ordinary differential equations”, or “ode’s”, because their solutions are functions of only one independent variable. We will also study “partial differential equations”, or “pde’s”, such as

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]

or

\[ \frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} \]

in which the unknown function, \( u(x,y) \) or \( y(x,t) \), depends on two or more independent variables. These will come in the last few weeks. For the moment we discuss only ode’s.

How do we solve ode’s?

Two important methods:

1. Use integration.

2. Guess.
Does every ode have a solution?

Yes.

Can we always find a formula for the solution?

NO!

\[ u'' = u^2 + e^u \]

Then how do I know there is a solution? What do we mean by a solution?

There are many important differential equations where we can’t find a formula for the solution. An example:

\[ y'' + y' + \sin y = 0 \]

This is the equation of motion for a pendulum swinging under the influence of gravity and with a damping term due to air resistance or friction. No formula is known for \( y(x) \). Yet, such a function must “exist”, for otherwise no pendulum could move!

Mathematicians can prove that there are functions which satisfy the ode, even if they can’t find a formula for such a function. Using a computer they can usually find an approximate graph of the function. We will learn to do this using Matlab.