Complex Variables, Summer 2016
Homework Assignments

Homeworks 1-4, due Thursday July 14th

Do twenty-four of the following problems.

Question 1

Let $a = 2 + i$ and $b = 1 - i$.
Sketch the complex numbers a, b, $a + b$, $a - b$, a^2, ab, $a\bar{b}$, b^2, ab^{-1} and ba^{-1}.
For each of these numbers, determine their modulus and argument.

Question 2

A complex number z obeys the relations $|z - 2 + i| = \sqrt{5}$ and $|z - 4i| = 4$.
What can we say about the number z?
In particular is z unique?
Explain your answer graphically.

Spiegel 1.71

Describe and graph the following curves in the complex plane:

- $|z - i| = 2$
- $|z + 2i| + |z - 2i| = 6$
- $|z - 3| - |z + 3| = 4$
- $z(\overline{z} + 2) = 3$
- $\Im(z^2) = 4$

Spiegel 1.73

Sketch the following regions in the complex plane and for each region discuss whether or not it is closed, open, bounded, unbounded, compact or non-compact.

- $1 < |z + i| \leq 2$
- $\Re(z^2) > 1$
- $|z + 3i| > 4$
- $|z + 2 - 3i| + |z - 2 + 3i| < 10$

Spiegel 1.85

An airplane travels 150 miles Southeast, 100 miles due West, 225 miles 30 degrees North of East and then 200 miles Northeast. Using complex numbers to represent each displacement, find the resulting position of the plane, relative to its starting position. Also sketch the trajectory of the airplane.

Spiegel 1.123

Let regions $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ in the complex plane be given as follows:

$$\mathbb{A} = \{z \in \mathbb{C} : |z + i| < 3\} \quad \mathbb{B} = \{z \in \mathbb{C} : |z| < 5\} \quad \mathbb{C} = \{z \in \mathbb{C} : |z + 1| < 4\}$$

Sketch $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ and the following sets:

- $\mathbb{A} \cap \mathbb{B} \cap \mathbb{C}$
- $\mathbb{A} \cup \mathbb{B} \cup \mathbb{C}$
- $\mathbb{A} \cap (\mathbb{B} \cup \mathbb{C})$
- $(\mathbb{A} \cup \mathbb{B}) \cap (\mathbb{B} \cup \mathbb{C})$

Question 7

Let α be a sixth root of 1 that is not real.
Show that either $\alpha^2 + \alpha + 1 = 0$, or $\alpha^2 - \alpha + 1 = 0$.
Show that α^2 is also a sixth root of 1 that is not real.
Also sketch all the possible complex numbers, α.

Question 8

Solve (algebraically) the equations $z^2 = -7+24i$ and $z^4 = -7+24i$ and plot the solutions on the complex plane.

Question 9

Find all square roots of $12i - 5$ and $12i + 5$ and plot the solutions.

Question 10

Find the equation of the circle that passes through the points i, $-2+3i$ and $-2+i$. Also find the equation of the tangent line to the circle, through the point $-2+3i$.

Question 11

Give the geometric interpretation of each of the following complex inequalities, with a sketch. Also discuss whether or not the set described by the solutions of the equation is open, closed, compact or non-compact and whether or not the solution set has interior points.

- $(3-i)z - (3+i)\overline{z} \neq 6i$
- $|z - 3i + 4|^2 \leq |z - 5|^2$
- $z\overline{z} - (4-3i)z - (4+3i)\overline{z} \leq 50$.

Question 12

Find all solutions of the equation $z^{10} = 1024$ and plot the solutions on the complex plane.

Spiegel Problems

Questions 13, 14, 15, 16, 17

From Spiegel, do the problems 1.109, 1.131, 1.146, 2.49, 2.50.

Question 18

Write the integer 1105 as a sum of squares: $1105 = a^2 + b^2$ where a and b are positive integers, with $a > b$ in four different ways.
Hint: begin by writing 1105 as a product of integer primes and then write each prime as $|z|^2$, for some complex number z.

Question 19

In quantum mechanics a spin state, α, describing a non-relativistic electron, or a qubit, is represented by a pair of complex numbers, $\alpha = (p, q)$ with p and q complex.

The size of a state $\alpha = (p, q)$, denoted $|\alpha|$, is then given by the formula $|\alpha| = \sqrt{|p|^2 + |q|^2}$.
The state is said to be normalized if its size is 1.

- Find the size of the state $\psi = (2 + i, 3 - 2i)$.

The states are treated like vectors: if $\beta = (r, s)$, with r and s complex numbers, is another state, then the sum $\alpha + \beta$ is computed termwise $(p, q) + (r, s) = (p + r, q + s)$.
Also if t is a complex number, we can scale α by a factor of t: the state $t\alpha = (tp, tq)$.

- Find a formula for all complex numbers t, such that the state $t\psi$ is normalized, where $\psi = (2 + i, 3 - 2i)$.

The states α and β are said to be orthogonal if and only if Pythagoras holds: $|t\alpha + \beta|^2 = |t|^2|\alpha|^2 + |\beta|^2$, for every complex number t.
Prove that $\alpha = (p, q)$ and $\beta = (r, s)$ are orthogonal if and only if $\overline{r}p + \overline{s}q = 0$.

- Find a formula for all states β orthogonal to the state $\psi = (2+i, 3-2i)$ and such that $|\beta| = 1$.

Question 20

The electric field at a point z of the complex plane, due to a charged line carrying charge q, perpendicular to the plane and passing through the plane at point w is $\dfrac{q}{\overline{z} - \overline{w}}$, where $z \neq w$.
The total electric field due to an assemblage of such charged lines is the sum of the individual fields.
One line of charge q passes through the origin, and two others, each carrying charge $4q$ pass through the points $4 + 3i$ and $4 - 3i$.
Where is the total field zero?

Question 21

Let $w = T(z) = (1 - 2i)z - 6 + 8i$, defined for any complex z.
Find the fixed point of T and a formula for the inverse transformation T^{-1}.
Also find the image under the transformation T in the w-plane of the following sets (with a sketch):

- $\{z : \Im(z) \geq 0\}$
- $\{z : \Re(z) = \Im(z)\}$
- $\{z : |z| = 5\}$.

Question 22

Let $w = \dfrac{1}{z}$, defined for any complex $z \neq 0$.

Find the image of the following sets (with sketch):

- $\{z : |z| = 2\}$
- $\{z : |z - 2i| = 2\}$
- $\{z : \Re(z) \geq 1\}$

Also prove that the image of a circle that passes through the origin is a straight-line that does not pass through the origin.

Question 23

Let $\alpha = (p, q)$ be a spin state, called a spinor, as in Question 18, with p and q complex numbers.
Associated to any spinor $\alpha = (p, q) \neq (0, 0)$ is a non-zero real vector in four dimensions:

$$N_\alpha = (t, x, y, z) = \left(|p|^2 + |q|^2,\ \ \overline{p}q + \overline{q}p,\ \ i(\overline{p}q - \overline{q}p),\ \ |p|^2 - |q|^2\right).$$

- Prove that N_α is a null (or light like) vector:

  $$t^2 - x^2 - y^2 - z^2 = 0.$$

  Note that N_α is also "future pointing" i.e. $t > 0$.
  Also prove that $N_{s\alpha} = |s|^2 N_\alpha$, for any non-zero complex number s.

- A boost with boost parameter r, a real number, in the (t, z)-direction (or Star-Trek transformation) is the linear transformation:

  $$V = (t,\ x,\ y,\ z) \to V_r = (\cosh(r)t + \sinh(r)z,\ x,\ y,\ \sinh(r)t + \cosh(r)z).$$

  Prove that if V is null, so is V_r.
  Prove that the spinor transformation $\alpha = (p,\ q) \to \alpha_r = \left(e^{\frac{r}{2}}p,\ e^{-\frac{r}{2}}q\right)$ produces a boost on the corresponding null vectors: if $N_\alpha = V$, then $N_{\alpha_r} = V_r$.

- A rotation in the (x, y) plane with angle θ is the linear transformation:

  $$V = (t,\ x,\ y,\ z) \to V_\theta = (t,\ \cos(\theta)x - \sin(\theta)y,\ \sin(\theta)x + \cos(\theta)y,\ z).$$

  Prove that if V is null, so is V_θ.
  Prove that the spinor transformation $\alpha = (p,\ q) \to \alpha_r = \left(e^{\frac{i\theta}{2}}p,\ e^{-\frac{i\theta}{2}}q\right)$ produces a rotation on the corresponding null vectors: if $N_\alpha = V$, then $N_{\alpha_\theta} = V_\theta$.

Question 24

Find the compositions $T_4 = T_2 \circ T_1$ and $T_3 = T_1 \circ T_2$ of the transformations

$$T_1 : z \to z^{-1}, \quad T_2 : z \to \frac{z}{z - 1}.$$

Give the domain of each of T_3 and T_4.
Show that T_4 and T_3 are inverses of each other.
Also determine the transformations T_4^2 and T_3^2.

Question 25

Find, with proof, a positive integer, n, which can be written in the form $n = a^2 + b^2$ with a and b positive integers, and $a < b$, in at least six different ways.

Question 26

Let $f(z) = iz^2 + \dfrac{1}{z}$.

Give the domain of $f(z)$ and write $f(z)$ as $u + iv$, where u and v are functions of the real variables x and y, with $z = x + iy$.
What is the common domain of the functions u and v?
Verify that the functions u and v obey the Cauchy-Riemann equations, $u_x = v_y$ and $u_y = -v_x$.

Question 27

Let $w = \dfrac{z + i}{z - i}$, defined for any complex $z \neq i$.

Find the image in the w-plane of the following sets:

- $\{z : \Im(z) \geq 0\}$
- $\{z : \Re(z) = \Im(z)\}$
- $\{z : |z| < 1\}$.

Question 28

Describe the transformation $T : z \to \dfrac{1}{1 - z}$ and determine the images under the transformations T and T^2 of the circle center $(4, -4)$ and radius 5 and of the line $x + y = 0$.
Also graph the curves and their images on one graph.

Question 29

Compute the following limits, or show that the limit in question does not exist

- $\lim_{z \to 4-3i} \frac{|z+3i-4|^2}{z^2-8z+25}$
- $\lim_{z \to -3i} \frac{z^4-81}{(z-2-3i)(z^2+9iz-18)}$

Question 30

Consider the function $f(z) = z^{\frac{1}{3}}$.
We choose $f(27) = 3$.

- If the cut for $f(z)$ is taken to be the the non-positive imaginary axis, determine the value of $f(-64)$.
- If, instead, the cut for $f(z)$ is taken to be the the non-negative imaginary axis, determine the value of $f(-64)$.
- If, instead, the cut for $f(z)$ is taken to be the the non-positive real axis, compare the values of the limits as $z \to -64$ of $f(z)$, where we approach from the upper and lower half-planes.

Question 31

Consider the function $f(z) = (z)^{\frac{1}{2}}(5-z)^{\frac{1}{2}}$.

- If its value at $z = 4$ is taken to be 2, with cuts along the positive real axis from 5 to infinity and along the negative real axis from 0 to minus infinity, determine the jump in its value as $z = 9$ is approached from above, vis à vis its value as $z = 9$ is approached from below.
- If, instead, its value at $z = 1+2i$ is taken to be $3+i$, with cuts along the real axis from 0 to 5 determine its value at $z = 9$.

Question 32

Consider the function $f(z) = (16 - z^2)^{\frac{1}{2}}$.

- If the value of $f(z)$ at $z = 3i$ is taken to be 5, with a cut along the real axis from -4 to 4, determine the jump in its value as $z = 0$ is approached from above, vis a vis its value as $z = 0$ is approached from below.

- If, instead, its value at $z = -5$ is taken to be $3i$, with vertical cuts in the upper half plane, one from -4, the other from 4, determine its values at $z = 5$ and at $z = 3i$.

Question 33

Consider the function $f(z) = \ln(z) - \ln(2 - z)$.

- If its value at $z = 1$ is taken to be 0, with cuts along the positive real axis from 2 to infinity and along the negative real axis from 0 to minus infinity, determine the jump in its value as $z = 6$ is approached from above, vis à vis its value as $z = 6$ is approached from below.

- If, instead, its value at $z = -2$ is taken to be $-\ln(2) - \pi i$, with a cut along the real axis from 0 to 2, determine its value at $z = 3$.

- Suppose that we want the function f to be well-defined and continuous on the circle centered at the origin, with radius 100.
  What does this tell us about the cuts that are allowed for the function f?
  Explain your answer.

Question 34

Determine (with proof) the following limits, or explain why the limit in question does not exist:

- $\lim_{z \to 1-i} \left(\frac{z^4 - 4}{z^3 - z^2 + iz^2 - z + 1 - i} \right)$.
- $\lim_{z \to -i} \left(\frac{z^2 - \overline{z}^2}{\overline{z} - i} \right)$.

Question 35

Determine, with proof the following quantities:

- All possible values of $(-i)^{-i}$
- All complex z, for which $\sin(z) = i$.

Question 36

Determine directly, from the definition of the derivative as a limit, the derivative of the function $f(z) = 2z(z^2 + 1)^{-1}$ and verify that $f'(z) = f_x = -if_y$.

Question 37

Show that $u(x, y) = x^3 + 6x^2y - 3xy^2 - 2y^3 + 2xy$ is harmonic and find its harmonic conjugate $v(x, y)$.
Write $f(x, y) = u(x, y) + iv(x, y)$ as a polynomial in the complex variable $z = x + iy$.

Question 38

Show that $u(x, y) = \frac{(x^2 - y^2 + 2xy)}{(x^2 + y^2)^2}$ is harmonic and find its harmonic conjugate $v(x, y)$.
Write $f(x, y) = u(x, y) + iv(x, y)$ as a function of the complex variable $z = x + iy$.