Notes on Minkowski space-time

Following Hermann Minkowski, the arena for special relativity is Minkowski space-time. The squared interval $S$ between points $X = (t, x, y, z)$ and $X' = (t', x', y', z')$ is:

$$S = c^2 (t - t')^2 - (x - x')^2 - (y - y')^2 - (z - z')^2.$$ 

Here $S$ has units of distance squared and $c$ is the speed of light. $t$ is a time coordinate and $x$, $y$ and $z$ are space coordinates.

If we use units of feet for distance and nano-seconds for time, $c$ is approximately 1, so one foot per nano-second ($10^{-9}$ seconds), actually a little less, with an error of less than two percent. Henceforth we take $c$ to be exactly 1.

The co-ordinates $X = (t, x, y, z)$ are called affine coordinates for Minkowski space-time. The numbers $t$, $x$, $y$ and $z$ range over the reals and the quadruple $(t, x, y, z)$ is called the position vector of the point $X$ relative to a given origin, the point $(0, 0, 0, 0)$.

- We say that the points $X$ and $X'$ are null related if $S = 0$. If $X$ and $X'$ are null related, then $X$ can communicate with $X'$ by a signal traveling at the speed of light, if $t < t'$ and on the other hand $X$ can receive a signal from $X'$ traveling at the speed of light if $t > t'$. Note that when $S = 0$, we can only have $t = t'$ if $X$ and $X'$ coincide.

- We say that the points $X$ and $X'$ are timelike related if $S > 0$. If they are timelike related, then $X$ can communicate with $X'$ by a signal traveling at less than the speed of light, if $t < t'$ and on the other hand $X$ can receive a signal from $X'$ traveling at less than the speed of light if $t > t'$. Note that when $S > 0$, we cannot have $t = t'$.

- We say that $X'$ is in the causal future of $X$ if $t < t'$ and $S \geq 0$.

- We say that $X'$ is in the causal past of $X$ if $t > t'$ and $S \geq 0$.

- We say that $X$ and $X'$ are acausal or space-like related if and only if $S < 0$. Such points cannot be connected by any known physical signal.
If we look at the case that the point $P'$ differs infinitesimally from the point $P$, we write the displacement $dX$ from $P$ to $P'$ as $dX = (dt, dx, dy, dz)$ and then the infinitesimal squared interval is written $ds^2$ and is called the metric, so here we have:

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2.$$ 

A curve is future pointing timelike if $ds^2 > 0$ and $dt > 0$. A curve is past pointing timelike if $ds^2 > 0$ and $dt < 0$. A curve is null if $ds^2 = 0$. A curve is spacelike if $ds^2 < 0$.

For a timeline curve the line integral of $ds = \sqrt{dt^2 - dx^2 - dy^2 - dz^2}$ along the curve is the proper time elapsed along the curve, which physically represents the aging along the curve.

**Example: the twin paradox**

Consider one twin, Artemis, at rest for $4\pi$ units of time. So if we take the origin as the starting point, we can model Artemis's trajectory as $x = y = z = 0$ and $0 \leq t \leq 4\pi$.

Then $dx = dy = dz = 0$, since $x, y$ and $z$ are constant, so the elapsed proper time $T_1$ in going from $(0, 0, 0, 0)$ to $(4\pi, 0, 0, 0)$ is:

$$T_1 = \int ds = \int_0^{4\pi} \sqrt{dt^2} = \int_0^{4\pi} dt = 4\pi.$$

Now consider another twin Babette, who begins and ends at the same space time point as does Artemis, but instead follows the space-time curve $t = 2u$, $x = \cos(u) - 1$, $y = \sin(u)$ and $z = 0$ for $u$ from 0 to $2\pi$.

So geometrically Babette traverses a circle in space, of radius one, in the $(x, y)$ plane, that passes through the origin. Then Babette has:

$$dt = 2du, \quad dx = -\sin(u)du, \quad dy = \cos(u)du, \quad dz = 0,$$

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = 4du^2 - \sin^2(u)du^2 - \cos^2(u)du^2 - 0 = 3du^2,$$

$$ds = \sqrt{3}du.$$

So the elapsed proper time $T_2$ in going from $(0, 0, 0, 0)$ to $(4\pi, 0, 0, 0)$ along this trajectory is:

$$T_2 = \int_0^{2\pi} \sqrt{3}du = 2\pi\sqrt{3}.$$
So the stay at home twin Artemis has aged relative to Babette by a factor of:

\[ \frac{T_2}{T_1} = \frac{2}{\sqrt{3}} = 1.154700539. \]

So Artemis has aged by about 15 percent more than has Babette.

**Note**

There is not really a paradox here; this effect is real and has been experimentally confirmed many times.

Another version of the paradox is when Babette goes out at uniform speed, turns around and immediately returns at uniform speed.

So, for example take Babette’s trajectory to be:

\[(u, au, 0, 0), \quad 0 \leq u \leq 1, \quad ds = \sqrt{1 - a^2} du\]

\[(v + 1, a - av, 0, 0), \quad 0 \leq v \leq 1, \quad ds = \sqrt{1 - a^2} dv.\]

Here \(|a| < 1\), since Babette’s trajectory is timelike.

Also Artemis’s trajectory is \((t, 0, 0, 0)\) for \(0 \leq t \leq 2\).

Again the twins start at the same point, the origin and end at the same point, the point \((2, 0, 0, 0)\).

Artemis ages by \(\int_0^2 dt = 2\) and Babette ages by:

\[\int_0^1 \sqrt{1 - a^2} du + \int_0^1 \sqrt{1 - a^2} dv = 2\sqrt{1 - a^2}.\]

So the aging of Babette relative to Artemis is: \(\sqrt{1 - a^2}\).

Note that the faster Babette travels the lower the relative aging.

Also in the limit that \(a \to 0\), the aging ratio goes to 1, as it should.

For example, if Babette travels at 0.96 of the speed of light, then if, when they meet again, Artemis has aged one hundred years, then Babette has aged only twenty eight years!

Another nice one is the case that \(a = 1 - 13^{-4} = \frac{28560}{28561} = 0.9999649872\), in which case the ratio is \(\frac{239}{13^4} = 0.008368054340\).

In this case Babette ages only 305.434 days whilst Artemis ages one hundred years. So if the twins were separated at birth, on their reunion, Babette is probably still not walking, whilst Artemis is old enough to be Babette’s great grand-parent!
Other co-ordinate systems

It is often useful to represent points in other coordinate systems, and then the metric has a corresponding form:

- **Cylindrical polar coordinates**: we put \( t = t, \ z = z, \ x = r \cos(\theta) \) and \( y = r \sin(\theta) \).

  Then we have:

  \[
  dt = dt, \quad dz = dz, \quad dx = \cos(\theta)dr - r \sin(\theta)d\theta, \quad dy = \sin(\theta)dr + r \cos(\theta)d\theta, \\
  dx^2 + dy^2 = (\cos(\theta)dr - r \sin(\theta)d\theta)^2 + (\sin(\theta)dr + r \cos(\theta)d\theta)^2 = dr^2 + r^2d\theta^2, \\
  ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = dt^2 - d\theta^2 - r^2d\phi^2.
  \]

- **Null coordinates**: we put \( t - z = u \) and \( t + z = v \), so the metric becomes \( ds^2 = du dv - dy^2 - dz^2 \).

- **Spherical polar coordinates**: we put \( t = t, \ z = \rho \cos(\theta), \ x = \rho \sin(\theta) \cos(\phi) \) and \( y = \rho \sin(\theta) \sin(\phi) \).

  Note that the cylindrical polar \( r \) is here \( \rho \sin(\theta) \). Then we have:

  \[
  dt = dt, \quad dz = \cos(\theta)d\rho - \rho \sin(\theta)d\theta \\
  dx = \rho \sin(\theta) \cos(\phi)d\rho + \rho \cos(\theta) \cos(\phi)d\theta - \rho \sin(\theta) \sin(\phi)d\phi, \\
  dy = \rho \sin(\theta) \sin(\phi)d\rho + \rho \cos(\theta) \sin(\phi)d\theta + \rho \sin(\theta) \cos(\phi)d\phi, \\
  dx^2 + dy^2 = d\rho^2 + \rho^2(\sin^2(\theta)d\phi^2 + \cos^2(\theta)d\phi^2), \\
  dx^2 + dy^2 + dz^2 = (\sin(\theta)d\rho + \cos(\theta)d\theta)^2 + \rho^2 \sin^2(\theta)d\phi^2 + (\cos(\theta)d\rho - \rho \sin(\theta)d\theta)^2 \\
  = d\rho^2 + \rho^2(d\theta^2 + \sin^2(\theta)d\phi^2). \\
  ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = dt^2 - d\rho^2 - \rho^2(d\theta^2 + \sin^2(\theta)d\phi^2).
  \]

- **Null spherical polars**: put \( t - \rho = u \) and \( t + \rho = v \).

  Then we have:

  \[
  ds^2 = dt^2 - d\rho^2 - \rho^2(d\theta^2 + \sin^2(\theta)d\phi^2) = du dv - \rho^2(d\theta^2 + \sin^2(\theta)d\phi^2).
  \]
Note that if someone handed us one of these later metrics or the same metric in some bizarre coordinate system, it might be quite hard to recognize that it is really just the ordinary Minkowski space-time in disguise. This illustrates the strategy of Albert Einstein: to Einstein a general gravitational field is represented by a general metric, so a quadratic in the differentials. If there is no gravity then, following Einstein, the metric has to agree with the Minkowski metric. We then need a mechanism, given the metric, for detecting the presence or absence of a gravitational field and for quantifying it. Again following Einstein, this mechanism is in the first instance the idea of geodesic deviation. Free particles in the Einstein theory follow timelike geodesics: curves of longest interval between any two points. If we have a bunch of particles, which initially are moving parallel to each other, if they stay parallel, then there is no gravity. If not there is a gravitational field and we say that they are acted on by a gravitational tidal force (usually attractive, so the trajectories tend to come together). The trick then is to invent a theory which determines these tidal forces.

Example
Consider the metric:

$$\frac{dudv - dx^2 - 2\cos(u)dxdy - dy^2}{1 - \cos(u - u_0)}.$$

Here \((u, v, x, y) \in \mathbb{R}^4\).
Also \(u_0\) is a constant and we restrict the range of the real variable \(u\) so that we have both \(-1 < \cos(u) < 1\) and \(\cos(u - u_0) < 1\).
So \(u \neq k\pi\) for any integer \(k\) and \(u \neq u_0 + 2m\pi\), for any integer \(m\).
Is this metric flat?
Symmetries: inertial observers

In Minkowski space-time an inertial observer is an observer moving along a timelike geodesic: these are straight lines with timelike tangent vector in the usual Minkowski co-ordinates. One principle of special relativity is that all these observers are on an equal footing, so there should be a symmetry group which maps one observer to another. This group is called the Poincaré group.

- If we represent a space-time point by the four-vector $X$, the simplest transformation is a displacement $X \rightarrow X+Y$, for some fixed four vector $Y$. This simply corresponds to a change of origin. The collection of all such transformations forms an abelian group the group of translations.

By such a transformation, we can imagine that any two observers share the same origin. Then the two observers are related by a linear transformation $X \rightarrow L(X)$, where $L$ preserves the metric of space-time: $L$ is called a Lorentz transformation. The collection of all such transformations forms the Lorentz group, a six dimensional group.

- Three of these dimensions are accounted for by rotations: if $X = (t, x, y, z)$, then a rotation preserves $t$ and rotates the $(x, y, z)$ coordinates, just as in three dimensional Euclidean space. A rotation depends on three parameters: two for the direction of the axis of rotation and one for the angle of rotation.

- The other three dimensions of the Lorentz group are accounted for by the relativistic analogue of the famous transformations of Galileo Galilei: to Galilleo, physics in a moving ship is represented by the transformation $(t, \vec{x}) \rightarrow (t, \vec{x} - vt)$, where $t$ is the time and $v$ is the velocity vector of the ship. The relativistic version is called a boost.

Modulo a rotation, we can consider the boost to be in the $(t, z)$ plane, so it is given by:

$$(t, x, y, z) \rightarrow (at + bz, x, y, ct + dz).$$

Here we need the relativistic square interval to be preserved, so we need:

$$t^2 - z^2 = (at + bz)^2 - (ct + dz)^2 = t^2(a^2 - c^2) + 2tz(ab - cd) - z^2(d^2 - b^2).$$

This formula needs to hold for all $t$ and $z$. Comparing coefficients, we get:

$$a^2 - c^2 = d^2 - b^2 = 1, \quad ab - cd = 0.$$
Then $a$ cannot be zero, so the equation $ab - cd = 0$ gives $b = kc$ and $d = ka$, for some $k$ (actually $k = da^{-1}$). Then we need $1 = d^2 - b^2 = k^2(a^2 - c^2) = k^2$. So $k = \pm 1$. If we take the time direction to be preserved, so $a > 0$, then we can write the transformation as:

$$(t, z) \rightarrow (\cosh(u)t - k \sinh(u)z), - \sinh(u)t + k \cosh(u)z).$$

Here $u$ is a real parameter. This reduces to the identity transformation when $u = 0$ provided $k = 1$, giving the final formula for the boost as:

$$(t, z) \rightarrow (\cosh(u)t - \sinh(u)z), - \sinh(u)t + \cosh(u)z).$$

Here $u$ is a real parameter. We put $v = \tanh(u)$, so $\cosh(u) = \frac{1}{\sqrt{1 - v^2}}$ and the boost transformation reads:

$$(t, z) \rightarrow \frac{1}{\sqrt{1 - v^2}}(t - vz, z - vt).$$

Here $|v| < 1$. In particular, an observer at rest at the spatial origin has $z = 0$ and $t = u$ a free parameter. This observer is mapped by the boost to the world-line:

$$(t, z) = \frac{1}{\sqrt{1 - v^2}}(u, -vu) = (s, -vs), \quad s = \frac{u}{\sqrt{1 - v^2}}.$$

In the co-ordinate system of the original observer this represents uniform motion with velocity $v$, just as in Galileo’s theory. However at equal time coordinates, so when the observer is at $(t, 0)$ with $t \geq 0$, and the boosted observer at $(t, -vt)$, the proper time for the original observer has elapsed an amount $t$, whereas the transformed observer has elapsed only $\sqrt{t^2 - v^2t^2} = \frac{t}{\sqrt{1 - v^2}}$ in proper time. The factor of $\sqrt{1 - v^2}$ is called the time dilation and leads directly to the twin paradox.

Summarizing the general Poincaré transformation has the form:

$$X \rightarrow L(X) + Y.$$

Here $Y$ is a constant vector and $L$ is a linear transformation acting on the vector $X$, such that $L(X).L(X) = X.X$ for any four-vector $X$, where $Z.Z$ is the Lorentzian quadratic form $t^2 - x^2 - y^2 - z^2$, for $Z = (t, x, y, z)$. Physically these are the general transformations preserving the (timeline geodesic) trajectories of inertial observers.