Vector spaces

We fix a field $\mathbb{F}$.
A vector space, $\mathbb{V}$, over the field $\mathbb{F}$, is a set $\mathbb{V}$, equipped with:

- An additive operation $+: \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{V}$, denoted $(v, w) \rightarrow v + w$, for any $v$ and $w \in \mathbb{V}$, which makes $\mathbb{V}$ into an abelian group with additive identity denoted $0 \in \mathbb{V}$ and additive inverse of $v \in \mathbb{V}$ denoted $-v$, for each $v \in \mathbb{V}$.

- An operation of scalar multiplication $\mathbb{F} \times \mathbb{V} \rightarrow \mathbb{V}$, denoted $(\lambda, v) \rightarrow \lambda v = v\lambda$, for each $\lambda \in \mathbb{F}$ and each $v \in \mathbb{V}$, such that, for any $v$ and $w$ in $\mathbb{V}$ and any $\lambda$ and $\mu$ in $\mathbb{F}$, we have the relations:
  
  $1v = v$, $0v = 0$, $(-1)v = -v$,

  $\lambda(v + w) = \lambda v + \lambda w$,

  $(\lambda + \mu)v = \lambda v + \mu v$,

  $\lambda(\mu v) = (\lambda\mu)v = \mu(\lambda)v$.

Examples of vector spaces

- The trivial vector space over a field $\mathbb{F}$ is a set with one element, denoted $0$, with the operations $0 + 0 = 0$ and $\lambda 0 = 0$, for each $\lambda \in \mathbb{F}$.

- The field $\mathbb{F}$ is a vector space over itself, with its usual operations.

- Let $S$ be a set.
  Then the space $\mathbb{F}^S$ of all maps from $S$ to $\mathbb{F}$ has the natural structure of a vector space, via the formulas, valid for each $f : S \rightarrow \mathbb{F}$, $g : S \rightarrow \mathbb{F}$ and for each $\lambda$ and $\mu$ in $\mathbb{F}$:

  $(\lambda f + \mu g)(s) = \lambda f(s) + \mu g(s)$, for each $s \in S$.

$\mathbb{F}^S$ is called the free vector space over $S$. 
Constructions of vector spaces

Subspaces

Given a vector space \( V \), and a subset \( X \) of \( V \), if \( X \) is invariant under the operations of \( V \), we may restrict these operations to \( X \) and then \( X \) becomes a vector space in its own right, called a subspace of \( V \).

To verify that \( X \subset V \) qualifies as a subspace, one need only show that \( \lambda x + \mu y \in X \), for any \( x \) and \( y \) in \( X \) and any \( \lambda \) and \( \mu \) in \( F \).

In general a linear combination of vectors of \( X \) is a vector \( x \) of the form \( x = \sum_{i=1}^{k} \lambda_i x_i \), for some \( x_i \in X, i = 1, 2, \ldots, k \), where \( \lambda_i \in F, i = 1, 2, \ldots, k \), for some positive integer \( k \). Then \( X \) is a subspace if and only it contains all linear combinations of vectors of \( X \).

The zero element of any vector space constitutes a subspace, called the trivial subspace.

The intersection of any family of subspaces of a vector space \( V \) is a subspace. If a set \( X \) is a subset of a vector space \( V \), then the intersection of all subspaces of \( V \) containing \( X \) is called \([X]\), the subspace spanned by \( X \). Then \([X] = [X] \) and \( X = [X] \) if and only if \( X \) is a subspace. The space \([X]\) consists of all possible linear combinations of the vectors of \( X \). Every vector space is spanned by itself. The dimension of a non-trivial vector space \( V \) is the smallest cardinality of a set that spans it. The trivial vector space is assigned dimension zero. A space \( V \) is called finite dimensional if it is spanned by a finite set: it then has dimension zero if and only if it is trivial and otherwise its dimension is a positive integer.

If \( X = \{x_1, x_2, \ldots, x_n\} \subset V \), then \([X]\) is written \([x_1, x_2, \ldots, x_n]\).

Then \([X]\) has dimension at most \( n \) and has dimension \( n \) if and only if the vectors \( x_j, j = 1, 2, \ldots, n \) are linearly independent:

\[
\sum_{j=1}^{n} \lambda_j x_j = 0 \text{ with } \lambda_j \in F, \text{ for each } j = 1, 2, \ldots, n, \text{ if and only if } \lambda_j = 0, \text{ for } j = 1, 2, \ldots, n.
\]

The vectors are linearly dependent if and only if they are not linearly independent, if and only if there is a relation \( 0 = \sum_{j=1}^{n} \lambda_j x_j = 0 \), with all \( \lambda_i \in F \) and at least one \( \lambda_k \) non-zero, if and only if the dimension of \([X]\) is less than \( n \).
If a collection of vectors \( \{x_1, x_2, \ldots, x_n\} \subset V \) spans a subspace \( \mathbb{X} \), then either \( \mathbb{X} = V \), or one can enlarge this collection to a list \( \{x_1, x_2, \ldots, x_n, x_{n+1}\} \subset V \), such that \( \mathbb{X} \subset [x_1, x_2, \ldots, x_{n+1}] \neq \mathbb{X} \): the new vector \( x_{n+1} \in V \) just has to be chosen to lie in the complement of \( \mathbb{X} \) in \( V \).

Then if the \( \{x_1, x_2, \ldots, x_n\} \) are linearly independent, so span a space of dimension \( n \), so are \( \{x_1, x_2, \ldots, x_{n+1}\} \) and they span a space of dimension \( n+1 \).

The collection of all \( n \)-dimensional subspaces of \( V \) is called the Grassmannian of all \( n \)-planes in \( V \), \( \text{Gr}(n, V) \).

If \( V \) has dimension \( m \), then \( \text{Gr}(n, V) \) is a space of dimension \( n(m-n) \).

Each one-dimensional subspace of \( V \) is a set \([x]\) of the form \([x] = \{\lambda x : \lambda \in \mathbb{F}\}\) for some \( 0 \neq x \in V \).

We have \([x] = [y] \), for \( 0 \neq y \in V \) if and only if \( x \) and \( y \) are linearly dependent.

The space of all one-dimensional subspaces of \( V \) is called \( \mathbb{P}(V) = \text{Gr}(1, V) \), the projective space of \( V \).

Each two-dimensional subspace of \( V \) is a set \([x, y]\) of the form \([x, y] = \{\lambda x + \mu y : \lambda \in \mathbb{F}, \mu \in \mathbb{F}\}\) for some linearly independent vectors \( x \) and \( y \) in \( V \).

We have \([x, y] = [p, q] \) if and only if there is an invertible matrix, \( \begin{vmatrix} a & b \\ c & d \end{vmatrix} \), with coefficients in \( \mathbb{F} \), such that \( p = ax + by, q = cx + dy \), if and only if each \( x \) and \( y \) are linearly independent, as are \( p \) and \( q \), but any three of the four vectors \( x, y, p \) and \( q \) are linearly dependent.

A subset \( \mathbb{X} \) of \( V \) is a subspace if and only if \([x, y] \subset \mathbb{X} \), for any \( x \) and \( y \) in \( \mathbb{X} \).
The direct sum of vector spaces

Let $S$ and $X$ be sets equipped with a surjective map $\pi : X \to S$, such that $X_s = \pi^{-1}(\{s\}) \subset S$ has the structure of a vector space over $F$, for each $s \in S$.

A section of $\pi$ is a map $f : S \to X$, such that $\pi \circ f = \text{id}_S$. Then the space $\Gamma$ of all sections of $\pi$ becomes a vector space by fibrewise addition and scalar multiplication: for any $f$ and $g$ in $\Gamma$, any $\lambda \in F$ and each $s \in S$:

$$(f + g)(s) = f(s) + g(s), \quad (\lambda f)(s) = \lambda f(s).$$

We write $\Gamma = \bigoplus_{s \in S} X_s$, called the direct sum of the family of vector spaces $X_s$. An element $f$ of $\Gamma$ is written $f = \bigoplus_{s \in S} f(s)$.

The support of a section $f \in \Gamma$ is the set of all $s \in S$ such that $f(s) = 0$. The collection of all elements of $\Gamma$ with finite support is a subspace of $\Gamma$, called the restricted direct sum of the family of spaces $\{X_s : s \in S\}$.

These two notions of direct sum coincide when $S$ is finite.

In particular if $X_s = F$ for each $s \in S$, then $\Gamma = F^S$.

Homomorphisms

Let $V$ and $W$ be vector spaces. Then a homomorphism or linear map, $T$ from $V$ to $W$ is a set map $T$ from $V$ to $W$, respecting the vector space structure:

$$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y),$$

for any $x$ and $y$ in $V$ and any $\lambda$ and $\mu$ in $F$.

Then the image $T(X)$ of a subspace $X$ of $V$ under $T$ is a subspace of $W$, since if $p$ and $q$ lie in $T(X)$ and if $\lambda$ and $\mu$ are in $F$, then $x$ and $y$ in $X$ exist, with $T(x) = p$ and $T(y) = q$, so then $z = \lambda x + \mu y$ lies in $X$, since $X$ is a subspace, so $T(z) = \lambda T(x) + \mu T(y) = \lambda p + \mu q \in T(X)$, so $T(X)$ satisfies the subspace criterion. In particular the image of $V$ itself is a subspace.

- $T$ is a monomorphism if and only if $T$ is injective.
- $T$ is a epimorphism if and only if $T$ is surjective.
- $T$ is an isomorphism if and only if $T$ is invertible, if and only if $T$ is both an epimorphism and a monomorphism.

In this case the inverse map $T^{-1}$ lies in $\text{Hom}(W, V)$. 


• $T$ is called an endomorphism in the case that $V = W$ and an automorphism if $T$ is an invertible endomorphism.

The automorphisms of $V$ form a group, under composition, called $\text{GL}(V)$.

The collection of all homomorphisms, $T : V \to W$ itself naturally forms a vector space denoted $\text{Hom}(V, W)$.

If $V, W$ and $X$ are vector spaces, then the composition map $\text{Hom}(V, W) \times \text{Hom}(W, X), (f, g) \to g \circ f$, for any $f \in \text{Hom}(V, W)$ and any $g \in \text{Hom}(W, X)$, is bilinear (i.e. linear in each argument).

In the particular case that $W = \mathbb{F}$, we put $V^* = \text{Hom}(V, \mathbb{F})$.

Then $V^*$ is called the dual space of $V$.

There is a natural homomorphism $J$ from $V \to (V^*)^*$ given by the formula, valid for any $v \in V$ and any $\alpha \in V^*$:

$$(Jv)(\alpha) = \alpha(v).$$

If $V$ is finite dimensional, then $J$ is an isomorphism.

A vector space $V$ over $\mathbb{F}$ is finite dimensional if and only if any one of the following three conditions is met:

• There is an epimorphism from $\mathbb{F}^P$ to $V$, for some finite set $P$

• There is a monomorphism from $V$ to $\mathbb{F}^Q$, for some finite set $Q$.

• There is an isomorphism from $\mathbb{F}^S$ to $V$ for some finite set $S$. 

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Matrix representations for linear transformations

If $S$ is a finite set put $\delta_s(t) = 0$, if $S \ni t \neq s$ and $\delta_s(s) = 1$.

Then $\delta_s \in F^S$ and for any $f \in F^S$, we have the unique expression $f = \sum_{s \in S} f(s) \delta_s$, so $\delta_S = \{\delta_s : s \in S\}$ forms a basis for $F^S$ called the canonical basis.

If $T$ is also a finite set and the canonical basis for $F^T$ is $\epsilon_T = \{\epsilon_t : t \in T\}$, then a homomorphism $M$ from $F^S$ to $F^T$ is given by the formulas:

$$M(\delta_s) = \sum_{t \in T} \epsilon_t M_{ts}, \text{ for each } s \in S,$$

$$M(f) = \sum_{t \in T} \sum_{s \in S} \epsilon_t M_{ts} f(s), \text{ for any } f \in F^S,$$

$$M(f)(t) = \sum_{s \in S} M_{ts} f(s), \text{ for any } f \in F^S \text{ and any } t \in T.$$

Here the matrix entry $M_{ts}$ is in $F$, for each $s \in S$ and $t \in T$.

Next, if $U$ is another finite set and $N$ a homomorphism from $F^T$ to $F^U$, where the canonical basis of $F^U$ is $\zeta_U = \{\zeta_u : u \in U\}$, then we have:

$$N(\epsilon_t) = \sum_{u \in U} \zeta_u N_{ut},$$

$$(N \circ M)(\delta_s) = \sum_{u \in U} \zeta_u (NM)_{us},$$

$$(NM)_{us} = \sum_{t \in T} N_{ut} M_{ts}.$$

This formula gives by definition the matrix product of the matrices representing $M$ and $N$.

If $e$ is an isomorphism of $F^S$ and a vector space $V$ and $f$ is an isomorphism of $F^T$ and a vector space $W$, then the set of vectors $E = \{e_s = e(\delta_s) : s \in S\}$ forms a basis for $V$, whereas the set of vectors $F = \{f_t = f(\epsilon_t) : t \in T\}$ forms a basis of $W$.

Then the map $M : \text{Hom}(V, W) \rightarrow \text{Hom}(F^S, F^T)$ given by the composition: $M(T) = f^{-1} \circ T \circ e$, for any $T \in \text{Hom}(V, W)$ is an isomorphism of vector spaces.
For $T \in \text{Hom}(\mathbb{V}, \mathbb{W})$ its image $M(T)$ is called the matrix of $T$ with respect to the bases $e$ and $f$.

We have the formula:

$$T(e_s) = \sum_{t \in T} f_t(M(T))_{ts},$$

for any $s \in S$ and any $T \in \text{Hom}(\mathbb{V}, \mathbb{W})$.

If $e'$ and $f'$ are also isomorphisms of $F^S$ and $F^T$ with $\mathbb{V}$ and $\mathbb{W}$, respectively, then we have:

$$M'(T) = (f')^{-1} \circ T \circ e' = ((f')^{-1} \circ f) \circ f^{-1} \circ T \circ e(e^{-1} \circ e') = FM(T)E^{-1}.$$

Here $F = (f')^{-1} \circ f$ is an automorphism of $F^T$ and $E = (e')^{-1} \circ e$, whose matrix representatives are invertible.
Tensor products

Let \( \mathbb{X} \) be a vector space over a field \( \mathbb{F} \). The tensor algebra of \( \mathbb{X} \), denoted \( \mathcal{T}(\mathbb{X}) \), is the associative algebra over \( \mathbb{F} \), generated by \( \mathbb{X} \) and \( \mathbb{F} \), such that the multiplication operations of the algebra are bilinear.

More explicitly, we may define a word \( w \) in \( \mathbb{X} \) of length \( k \), a nonnegative integer, to be an ordered \((k+1)\)-tuple \( w = \{ \lambda, w_1, w_2, \ldots, w_k \} \), where \( \lambda \in \mathbb{F} \) and each \( w_i, i = 1, 2, \ldots, k \) lies in \( \mathbb{X} \). If also \( x = \{ \mu, x_1, x_2, \ldots, x_m \} \) is a word of length \( m \), the product \( z = wx \) is the word of length \( k + m \), given by \( z = \{ \nu, z_1, z_2, \ldots, z_{m+k} \} \), where \( \nu = \lambda \mu \in \mathbb{F} \), \( z_i = w_i \in \mathbb{X} \), for \( 1 \leq i \leq k \) and \( z_i = x_{i-k} \in \mathbb{X} \), for \( k + 1 \leq i \leq k + m \). Then we have \((xy)z = x(yz)\), for all words \( x, y \) and \( z \) in \( \mathbb{X} \). The tensor algebra \( \mathcal{T}(\mathbb{X}) \) is then the span of all the words, subject to the conditions than the multiplication of words is bilinear in its arguments and to the requirement that if \( w = \{ \lambda, w_1, w_2, \ldots, w_k \} \) is a word and \( w' = \{ \lambda', w'_1, w'_2, \ldots, w'_k \} \) is another word of the same length, then \( w = w' \) if \( w'_i = \lambda_i w_i \), for all \( i \) and some \( \lambda_i \in \mathbb{F} \), where \( \lambda' \Pi_{i=1}^{k} \lambda'_i = \lambda \).

Note that the word \( w = \{ \lambda, w_1, w_2, \ldots, w_k \} \) is then the (ordered) product of the word \( \{ \lambda \} \) of length zero and the words \( \{ 1, w_i \} \) of length one. The words of length zero may be identified with the field \( \mathbb{F} \) and the words of length one with the vector space \( \mathbb{X} \). Also note that any word with one or more entries zero gives the zero element of the tensor algebra. We abbreviate the words \( w, x \) and \( z = wx \) as \( w = \lambda w_1 w_2 \ldots w_k \), \( x = \mu x_1 x_2 \ldots x_m \) and \( z = wx = \lambda \mu w_1 w_2 \ldots w_k x_1 x_2 \ldots x_m \).

By definition, for each non-negative integer \( k \), \( \mathcal{T}^k(\mathbb{X}) \) is the subspace of \( \mathcal{T}(\mathbb{X}) \) spanned by the words of length \( k \). Then the algebra product maps \( \mathcal{T}^k(\mathbb{X}) \times \mathcal{T}^m(\mathbb{X}) \) into \( \mathcal{T}^{k+m}(\mathbb{X}) \), for any non-negative integers \( k \) and \( m \) (where \( \mathcal{T}^0(\mathbb{X}) = \mathbb{F} \) and \( \mathcal{T}^1(\mathbb{X}) = \mathbb{X} \)).

The full tensor algebra of \( \mathbb{X} \), is the quotient of \( \mathcal{T}(\mathbb{X} \oplus \mathbb{X}^*) \) by the relations \( \alpha v = v \alpha \), for any \( v \in \mathcal{T}(\mathbb{X}) \) and any \( \alpha \in \mathcal{T}(\mathbb{X}^*) \). In this algebra, the subspace spanned by the words \( \alpha w \), with \( \alpha \in \mathcal{T}^q(\mathbb{X}^*) \) and with \( w \in \mathcal{T}^p(\mathbb{X}) \) is denoted \( \mathcal{T}_q^p(\mathbb{X}) \) and we have that the product maps \( \mathcal{T}_q^p(\mathbb{X}) \times \mathcal{T}_s^r(\mathbb{X}) \) into \( \mathcal{T}_{q+r}^{p+s}(\mathbb{X}) \), for any non-negative integers \( p, q, r \) and \( s \). When \( \mathbb{X} \) is finite dimensional, \( \mathcal{T}_q^p(\mathbb{X}) \) and \( \mathcal{T}_p^q(\mathbb{X}^*) \) are naturally isomorphic, for any nonnegative integers \( p \) and \( q \).

The words spanning \( \mathcal{T}_q^p(\mathbb{X}) \) may be written \( \lambda w_1 w_2 \ldots w_p \alpha_1 \alpha_2 \ldots \alpha_q \), where \( \lambda \in \mathbb{F}, w_1, w_2, \ldots, w_p \) lie in \( \mathbb{X} \) and \( \alpha_1, \alpha_2, \ldots, \alpha_q \) lie in \( \mathbb{X}^* \).
If the vector space $\mathbb{X}$ has dimension $n$ and if $\{e_i, i = 1, 2, \ldots, n\}$ is a set of vectors of $\mathbb{X}$ that spans $\mathbb{X}$, so forms a basis, then any tensor $\tau$ of type $(p,0)$ can be written uniquely:

$$\tau = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_p=1}^{n} \tau^{i_1i_2\ldots i_p} e_{i_1} e_{i_2} \ldots e_{i_p}.$$ 

If also $\{e^i, i = 1, 2, \ldots, n\}$ is the dual basis of $\mathbb{X}^*$, so we have $e^i(e_i) = \delta^i_i$, for any $1 \leq i \leq n$ and $1 \leq j \leq n$, then any tensor $\upsilon$ of type $(0,q)$ can be written uniquely:

$$\upsilon = \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_q=1}^{n} \upsilon^{j_1j_2\ldots j_q} e_{j_1} e_{j_2} \ldots e_{j_q}.$$ 

Also any tensor $\phi$ of type $(p,q)$ can be written uniquely:

$$\phi = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_p=1}^{n} \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_q=1}^{n} \phi^{i_1i_2\ldots i_p}_{j_1j_2\ldots j_q} e_{i_1} e_{i_2} \ldots e_{i_p} e_{j_1} e_{j_2} \ldots e_{j_q}.$$ 

Here each of the coefficients $\tau^{i_1i_2\ldots i_p}$, $\upsilon^{j_1j_2\ldots j_q}$ and $\phi^{i_1i_2\ldots i_p}_{j_1j_2\ldots j_q}$ lies in $\mathbb{F}$ and each coefficient may be chosen arbitrarily. 

In particular, it follows that the vector space of tensors of type $(p,q)$ is isomorphic to $\mathbb{F}^r$, where $r = n^{p+q}$.

Finally, if $\{A_s : s \in \mathbb{S}\}$ is a set of subspaces of $\mathbb{X}$ labelled by a totally ordered finite set $\mathbb{S}$, then the tensor product $\otimes_{s \in \mathbb{S}} A_s$ is the subspace of the tensor algebra of $\mathbb{X}$ spanned by the words formed from the ordered product of the elements of the image of maps $x : \mathbb{S} \to \mathbb{X}$, such that $x(s) \in A_s$, for all $s \in \mathbb{S}$. 

9
The Kronecker delta and the trace

Let $V$ and $W$ be vector spaces over $\mathbb{F}$. The vector space $V^* \otimes W$ acts naturally on $V$, such that for any $\alpha \in V^*$, any $v \in V$ and $w \in W$, we have:

$$(\alpha \otimes w)(v) = \alpha(v)w.$$ 

This gives an element of $\text{Hom}(V, W)$.

This induces a natural map, easily seen to be an endomorphism, from $V^* \otimes W$ to $\text{Hom}(V, W)$. This map is an isomorphism if $V$ is finite dimensional.

When $V$ and $W$ are both finite dimensional, the following spaces are all naturally isomorphic:

- $\text{Hom}(V, W)$
- $V^* \otimes W$
- $W \otimes V^*$
- $\text{Hom}(W^*, V^*)$

Also the dual space of this quartet of spaces has four corresponding versions:

- $\text{Hom}(W, V)$
- $W^* \otimes V$
- $V \otimes W^*$
- $\text{Hom}(V^*, W^*)$

In particular, if $V = W$, we see that $\text{Hom}(V, V)$ is its own dual. This entails that there is a natural map $E : \text{Hom}(V, V) \times \text{Hom}(V, V) \to \mathbb{F}$, which is bilinear in its arguments. Specifically we have the formulas, valid for any $\alpha$ and $\beta$ in $V^*$, $v$ and $w$ in $V$ and any $T$ in $\text{Hom}(V, V)$:

$$E(\alpha \otimes v, \beta \otimes w) = \beta(v)\alpha(w),$$
$$E(\alpha \otimes v, T) = \alpha(T(v)).$$

Then $E$ is a non-degenerate symmetric bilinear form on $\text{Hom}(V, V)$: for any $S$ and $T$ in $\text{Hom}(V, V)$, we have $E(S, T) = E(T, S)$ and if $S \in \text{Hom}(V, V)$, then $E(S, W) = 0$, for all $W \in \text{Hom}(V, V)$, if and only if $S = 0$. 

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There is a distinguished element of $\text{Hom}(V,V)$, the identity operator $\delta_V$, defined by the formula, valid for any $v \in V$.

$$\delta_V(v) = v.$$  

Then $\delta_V$ is called the Kronecker delta tensor for $V$. Then $\delta_V$ gives a natural element of $\text{Hom}(V,V)$, called $\text{tr}_V$, the trace, given by the formula, valid for any $T \in \text{Hom}(V,V)$:

$$\text{tr}_V(T) = E(\delta_V, T).$$

In particular we have, for any $\alpha \in V^*$ and any $v \in V$ the formula:

$$\text{tr}_V(\alpha \otimes v) = \alpha(v).$$

Also we have, for any $S$ and $T$ in $\text{Hom}(V,V)$:

$$\text{tr}_V(S \circ T) = \text{tr}_V(S \circ T).$$

Finally we have a key formula:

$$\text{tr}_V(\delta_V) = n.$$  

Here $n$ is the dimension of $V$, but here regarded as an element of $\mathbb{F}$, not as a positive integer.

If $X$ is a vector space with dual $X^*$, the full tensor algebra of $X$ is the tensor algebra of $X \oplus X^*$. It is usual to quotient this algebra by the relations $\alpha w = w \alpha$, for any word $w$ in $X$ and any word $\alpha$ in $X^*$. This gives the algebra of mixed tensors of $X$. If a word of this algebra is the concatenation of $p$ letters form $X$ and $q$ from $X^*$, the word is said to be of type $(p,q)$. The words of type $(p,0)$ are said to be contravariant, the words of type $(0,q)$ covariant. Then $T^p_q(X)$ is the subspace of the tensor algebra spanned by words of type $(p,q)$. Under the tensor product, we have $T^p_q(X) \otimes T^s_r(X) \subset T^{p+s}_{q+r}(X)$, for any nonnegative integers $p,q,r$ and $s$ (Here $T^0_0(X) = \mathbb{F}$). Note that $T^p_0(X) = T^0_p(X)$ and $T^q_0(X) = T^q(X) \otimes T^0(X^*)$ and $T^0_q(X) = T^0(X)$, whereas $T^0_q(X) = T^q(X^*)$, for all non-negative integers $p$ and $q$.  

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For each positive integer \( s \), denote by \( \mathbb{N}_s \) the (totally ordered) set \( \mathbb{N}_s = \{ t \in \mathbb{N} : 1 \leq t \leq s \} \). Now let \( \tau \) be a word in \( T_{pq}(X) \), so \( \tau \) has the form \( \tau = x \otimes \alpha \), where \( x = \otimes_{k \in \mathbb{N}_p} x_k \) and \( \alpha = \otimes_{j \in \mathbb{N}_q} \alpha_j \) and each \( x_k, k = 1, 2, \ldots, p \) is in \( X \), whereas each \( \alpha_j, j = 1, 2, \ldots, q \) is in \( X^* \). For some integer \( r \) with \( 1 \leq r \leq \min(p, q) \), let there be given a bijection \( \mu \) from a subset \( A_r \) of \( \mathbb{N}_p \) with \( r \) elements to a subset \( B_r \) of \( \mathbb{N}_q \), also with \( r \) elements. Then define \( \text{tr}_\mu(\tau) \in T_{q-r}^{p-r}(X) \) by the formula:

\[
\text{tr}_\mu(\tau) = \lambda x' \otimes \alpha',
\]

\[
x' = \otimes_{k \in \mathbb{N}_p - A_r} x_k \in T_0^{p-r}(X),
\]

\[
\alpha' = \otimes_{j \in \mathbb{N}_q - B_r} \alpha_j \in T_0^q(X),
\]

\[
\lambda = \prod_{i=1}^{r} \alpha_{\mu(i)}(x_{\mu(i)}) \in \mathbb{F}.
\]

Note that the maps \( \mu \) and \( \mu^{-1} \) are not, in general, order preserving. Then \( \text{tr}_\mu \) extends naturally to a linear map, still called \( \text{tr}_\mu \), from \( T_{pq}(X) \) to \( T_{q-r}^{p-r}(X) \): it is called the \( \mu \) trace applied to \( T_{pq}(X) \). In the special case that \( p = q = r = 1 \), the map \( \mu \) is the identity and \( \text{tr}_\mu \) equals the ordinary trace \( \text{tr}_X \), defined above.

**Tensors in the one-dimensional case**

Let \( L \) be a one-dimensional space over a field \( \mathbb{F} \). Then the tensor algebra of \( L \) is abelian. For any positive integer \( r \), denote by \( L^r \) the \( r \)-fold tensor product of \( L \) with itself, with, in particular, \( L^1 = L \). Then the dual space of \( L \) may be denoted \( L^{-1} \) and for any positive integer \( s \), its \( s \)-fold tensor product with itself may be denoted \( L^{-s} \). Finally we put \( L^0 = \mathbb{F} \). Then there is a natural isomorphism of \( L^r \otimes L^s \) with \( L^{r+s} \), for any integers \( r \) and \( s \). If \( \{ e \} \) is a basis for \( L \), the induced basis for \( L^r \) is denoted \( \{ e^r \} \), for any integer \( r \). In particular \( e^0 = 1 \) and \( e^{-1} \) is dual to \( e \), \( e^{-1}e = 1 \) and \( \{ e^{-1} \} \) is a basis for \( L^{-1} \). Then a general tensor \( T \) can be written uniquely in the form:

\[
T = \sum_{r \in \mathbb{Z}} t_re^r.
\]

Here the map \( \mathbb{Z} \to \mathbb{F}, r \to t_r \) is required to be non-zero for only a finite number of integers \( r \).

The (associative, commutative) tensor multiplication of \( T \) with the tensor \( U = \sum_{r \in \mathbb{Z}} u_se^s \) is the tensor:

\[
TU = UT = \sum_{n \in \mathbb{N}} v_n e^n, \quad v_n = \sum_{r+s=n} t_r u_s.
\]
The rank theorem for homomorphisms

Let \( V \) and \( W \) be finite dimensional vector spaces over \( F \), with dimensions \( m \) and \( n \), respectively and let \( T : V \to W \) be a linear transformation. Associated to \( T \) is its image, \( T(V) \), which we have proved above to be a subspace of \( W \).

Also associated to \( T \) is its kernel \( \ker(T) = \{ v \in V : T(v) = 0 \} \).

If \( v \) and \( w \) are in \( \ker(T) \), so \( T(v) = T(w) = 0 \), then for any \( \lambda \) and \( \mu \) in \( F \), we have:
\[
T(\lambda v + \mu w) = \lambda T(v) + \mu T(w) = 0.
\]
So \( \lambda v + \mu w \in \ker(T) \).

So \( \ker(T) \) is a subspace of \( V \). \( T \) is a monomorphism if and only if \( \ker(T) = 0 \).

Let \( E_2 = \{ e_\alpha, \alpha \in S \} \) form a basis of \( \ker(T) \) (where \( E_2 \) is empty if \( \ker(T) = \{ 0 \} \)).

This has at most \( m \) elements, since any \( m + 1 \) vectors of \( V \) are linearly dependent.

Since any \( n + 1 \) vectors of \( W \) are linearly dependent, the same is true of any subspace of \( W \), so \( T(V) \) has finite dimension \( k \), for some non-negative integer \( k \leq n \).

Let \( \{ f_1, f_2, \ldots, f_k \} \in T(V) \) be a minimal spanning set for \( T(V) \) (regarded as the empty set if \( T = 0 \)).

Then by repeatedly adding new vectors \( f_{k+1}, f_{k+2}, \ldots \), whilst maintaining linear independence, we can extend to a list \( F = \{ f_1, f_2, \ldots, f_n \} \), yielding a basis for \( W \).

Let \( e_i \in V \) obey the relations \( T(e_i) = f_i \), for each \( i = 1, 2, \ldots k \).

Then if \( \lambda_i \in F \) obey the relation \( \sum_{i=1}^{k} \lambda_i e_i = 0 \), applying \( T \) to this equation, we get:
\[
0 = T(0) = T \left( \sum_{i=1}^{k} \lambda_i e_i \right) = \sum_{i=1}^{k} \lambda_i T(e_i) = \sum_{i=1}^{k} \lambda_i f_i.
\]

But the \( \{ f_i; i \in \mathbb{N}_k \} \) are linearly independent in \( W \), so all the \( \lambda_i \) must vanish, for \( i \in \mathbb{N}_k \).

So the set \( E_1 = \{ e_i; i \in \mathbb{N}_k \} \) is a linearly independent set of vectors in \( V \).
Now consider the union $E$ of the sets $E_1$ and $E_2$.
If $\sum_{i=1}^k \lambda_i e_i + \sum_{\alpha \in S} \mu_\alpha e_\alpha = 0$, for some $\{\lambda_i : i \in \mathbb{N}_k\}$ and $\{\mu_\alpha : \alpha \in S\}$, then applying $T$, we get the formula:

$$0 = T(0) = T\left( \sum_{i=1}^k \lambda_i e_i + \sum_{\alpha \in S} \mu_\alpha e_\alpha \right)$$

$$= \sum_{i=1}^k \lambda_i T(e_i) + \sum_{\alpha \in S} \mu_\alpha T(e_\alpha)$$

$$= \sum_{i=1}^k \lambda_i f_i + \sum_{\alpha \in S} \mu_\alpha (0) = \sum_{i=1}^k \lambda_i f_i.$$

But the $\{f_i : i \in \mathbb{N}_k\}$ is linearly independent.
So the $\lambda_i, i = 1, 2, \ldots, k$ are all zero.
Back substituting, we get: $0 = \sum_{\alpha \in S} \mu_\alpha e_\alpha$.
But the $\{e_\alpha : \alpha \in S\}$ are linearly independent.
So all the $\mu_\alpha$ vanish, for each $\alpha \in S$.
So the set $E$ is a linearly independent set.
Next let $v \in V$.
Then $T(v) \in T(V)$, so there is an expression:

$$T(v) = \sum_{i=1}^k v_i f_i,$$
for some $v_i \in \mathbb{F}, i = 1, 2, \ldots, k$.

Put $w = v - \sum_{i=1}^k v_i e_i \in V$.
Then we have:

$$Tw = T\left( v - \sum_{i=1}^k v_i e_i \right) = Tv - \sum_{i=1}^k v_i T(e_i)$$

$$= Tv - \sum_{i=1}^k v_i f_i = 0.$$

So $w \in \ker(T)$, so we have an expression $w = \sum_{\alpha \in S} w_\alpha e_\alpha$, for some $\{w_\alpha \in \mathbb{F} : \alpha \in S\}$. Then we have the formula:

$$v = w + \sum_{i=1}^k v_i e_i = \sum_{\alpha \in S} w_\alpha e_\alpha + \sum_{i=1}^k v_i e_i.$$
So the set $E$ is linearly independent and spans $\mathbb{V}$, so is a basis for $\mathbb{V}$.
In particular the cardinality of $S$ is $m - k$ and we have proved the formula:

$$\nu(T) + \rho(T) = \dim(\mathbb{V}).$$

Here $\rho(T)$, called the rank of $T$ is the dimension of the image of $T$ and $\nu(T)$, called the nullity of $T$ is the dimension of the kernel of $T$.

Now by relabeling the elements of $E_2$, we may write $E = \{e_i : i \in \mathbb{N}_m\}$.

Now we have the Rank Theorem in two versions:

- With respect to the bases $E$ of $\mathbb{V}$ and $F$ of $\mathbb{W}$, the matrix $M$ of $T$ is:

  $$T(e_i) = \sum_{j=1}^{n} f_j M_{ji}, \text{ for any } i \in \mathbb{N}_m.$$ 

  Here we have $T(e_i) = 0$, for $i > k$ and $T(e_i) = f_i$, for $1 \leq i \leq k$, so the matrix $M_{ji}$ has the block form:

  $$\begin{pmatrix}
  I_k & 0 \\
  0 & 0 
  \end{pmatrix}.$$

  Here $I_k$ is the $k \times k$ identity matrix.

- More geometrically, we can decompose $\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_2$ and $\mathbb{W} = \mathbb{W}_1 \oplus \mathbb{W}_2$, where $T$ gives an isomorphism of $\mathbb{V}_1$ with $\mathbb{W}_1 = T(\mathbb{V})$, both spaces of dimension $k$ and $T$ is zero on $\mathbb{V}_2 = \ker(T)$.

A corollary is that given any $n \times m$-matrix $M$, there is an invertible $n \times n$-matrix $E$ and an invertible $m \times m$ matrix $F$, such that $EMF^{-1} = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$, for some non-negative integer $k \leq \min(m, n)$ and the number $k$, called the rank of the matrix $M$, is independent of the choice of the matrices $E$ and $F$.  

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Monomorphisms, epimorphisms and morphisms

We first describe monomorphisms and epimorphisms, in term of each other.

- If \( V \) and \( W \) are vector spaces, a map \( L : V \to W \) is a monomorphism if and only if there exists a vector space \( X \) and a linear map \( M : W \to X \), such that \( ML \) is an isomorphism.

- If \( W \) and \( X \) are vector spaces, a linear map \( M : W \to X \) is an epimorphism if and only if there exists a vector space \( V \) and a linear map \( L : V \to W \), such that \( ML \) is an isomorphism.

Given vector spaces \( V, W \) and \( X \), \( L \in \text{Hom}(V, W) \) and \( M \in \text{Hom}(W, X) \), we say that the pair \( (L, M) \) forms a dual pair if and only if \( ML \) is an isomorphism. If \( (L, M) \) is a dual pair, then \( L \) is necessarily a monomorphism and \( M \) is necessarily an epimorphism. So given \( L \in \text{Hom}(V, W) \), \( L \) is a monomorphism if and only if \( X \) and \( M \in \text{Hom}(W, X) \) exist such that \( (L, M) \) is a dual pair. Conversely, given \( M \in \text{Hom}(W, X) \), \( M \) is an epimorphism if and only if \( V \) and \( L \in \text{Hom}(V, W) \) exist such that \( (L, M) \) is a dual pair.

Note that if \( L \) is a monomorphism from \( V \) to \( W \), then there is a dual pair \( (L, M) \) with \( M : W \to X \) and \( ML = \text{id}_V \). Also if \( M \) is an epimorphism from \( W \) to \( V \), then there is a dual pair \( (L, M) \) with \( L : V \to W \) and \( ML = \text{id}_V \).

Now let \( T : X \to Z \) be any homomorphism from the vector space \( X \) to the vector space \( Z \). Then there exists a factorization \( T = LM \), with \( M \) an epimorphism from \( X \) to a vector space \( Y \) and \( L \) a monomorphism from \( Y \) to \( Z \). Also there exist a monomorphism \( S \) from \( Y \) to \( X \) and an epimorphism \( U \) from \( Z \) to \( Y \), such that \( UTS \) is an isomorphism.

If \( T = LM = L'M' \) with \( M : X \to Y \), \( M' : X \to Y' \), \( L : Y \to Z \) and \( L' : Y' \to Z \) and with \( L \) and \( L' \) monomorphisms, whereas \( M \) and \( M' \) are epimorphisms, then there is an isomorphism \( N : Y \to Y' \), such that \( L'N = L \) and \( M' = NM \). In particular the dimension of \( Y \) is independent of all choices. It is called the rank of \( T \) and is bounded by the min(\( \dim(X), \dim(Z) \)), since if \( \dim(Y) > \dim(Z) \), there is no monomorphism from \( Y \) to \( Z \) and if \( \dim(Y) > \dim(X) \), there is no epimorphism from \( X \) to \( Y \). The rank of \( T \) is equal to \( \dim(X) \) if and only if \( T \) is a monomorphism and the rank of \( T \) is equal to \( \dim(Z) \) if and only if \( T \) is an epimorphism.
Pictures

We represent vectors and tensors by pictures:

This represents a vector of a vector space \( V \).
The line represents that the vector space in question is \( V \).
For another vector in the same space, we can use a different shape:

Addition and scalar multiplication can be represented naturally:

Depending on the context it might be more appropriate to bring the lines together, to emphasize that they represent only a single vector:

Also, depending on the context it might be more appropriate to tie the scalars to the vector that they act on, in which case, if the lines are brought together, the plus sign becomes superfluous:
Alternatively, we can tie the scalar to the vector:

If we have a vector of another vector space $\mathcal{W}$, we use a different colour:

If we have an element of the dual vector space $\mathcal{V}^*$, it is represented by a downward oriented line:

Objects without lines emerging from them represent scalars, or elements of the field. So for example the dual pairing of a co-vector and a vector is represented by joining the two:

Tensors based on $\mathcal{V}$ and $\mathcal{V}^*$ are represented by multiple lines:

- A tensor of type $(3, 0)$:

- A tensor of type $(0, 3)$:
- A tensor $T$ of mixed type $(3, 3)$:

![Tensor Diagram]

Tensors act on bunches of vectors and dual vectors, as appropriate to get scalars. So, for example, when the $(3, 3)$ tensor $T$ acts on three vectors $v$, $w$ and $x$ in $V$ and three dual vectors $\alpha$, $\beta$ and $\gamma$ in $V^*$, we get the scalar:

This scalar, usually written $T(v, w, x, \alpha, \beta, \gamma)$ is linear in all seven arguments.

Sometimes we need to permute the arguments of tensors, so for example if we permute the first and third vector arguments and cyclically permute the dual vector arguments of $T$, we can create the following tensor $U$:

![Permuted Tensor Diagram]

In general the tensor $U$ is not the same tensor as $T$, since its action on the same vector and dual vector arguments is different.

The (associative) tensor product is just denoted by juxtaposition:

- A tensor product $T \otimes U$:
Here the tensor product of the $(3, 0)$ tensor $T$ and the $(2, 0)$ tensor $U$ is the $(5, 0)$ tensor $T \otimes U$.

- A mixed tensor product $T \otimes W$:

\[ T \otimes W \]

This tensor product produces a tensor $T \otimes W$ of mixed type $(3, 2)$.

- It is usual when considering mixed tensors, to assume that tensors based on $\mathbb{V}$ commute with those based on $\mathbb{V}^*$, so we can also write $T \otimes W$, without ambiguity as, for example:

\[ W \otimes T \]

- The contraction of tensors is represented by joining up the lines. So, for example, if we contract the third argument of $T$ with the first argument of $W$ and the second argument of $T$ with the second argument of $W$, we get a dual vector:

\[ T \otimes W \]

Notice in particular that the action of the $(3, 3)$-tensor $T$ on the vectors $v$, $w$ and $x$ and the dual vectors $\alpha$, $\beta$ and $\gamma$ given above is a complete contraction (meaning that the result is a scalar) of the tensor product $T \otimes \alpha \otimes \beta \otimes \gamma \otimes v \otimes w \otimes x$. 

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An endomorphism $L$ of a vector space is represented as a $(1,1)$ tensor:

This acts in many different ways, every one linear in each argument, for example:

- Mapping a vector $v$ to a vector $L(v)$:

- Mapping a dual vector $\alpha$ to a dual vector $\alpha(L)$:

- Mapping a vector $v$ and a dual vector $\alpha$ to a scalar $\alpha(L(v))$:
Mapping an endomorphism $M$ to a scalar $\text{tr}(LM)$:

The Kronecker delta tensor, the identity endomorphism, is just represented by a line, so we have the trace of $L$ as:

The trace of the Kronecker delta is $n$ the dimension of the vector space, regarded as an element of the field of scalars $\mathbb{F}$, not as an integer:
The elements of the direct sum of two (or more) vector spaces are represented by the merging of their colors to form a new color:

Here the blue/green and the yellow/green endomorphism are injections of the blue and yellow vector spaces into the green one, respectively.

An \((L, M)\) pair, with \(L : \mathbb{V} \to \mathbb{W}\) an injection and \(M : \mathbb{W} \to \mathbb{V}\) a surjection, such that \(ML = \text{id}_\mathbb{V}\) is written as follows:

\[
L = \begin{matrix} \square \\ \end{matrix}, \quad M = \begin{matrix} \square \\ \end{matrix},
\]
A \mathcal{V} basis for a vector space \mathcal{W} is represented by a pair of isomorphisms, the basis and the dual basis:
These are required to compose both ways to give the Kronecker delta:

\[
\begin{align*}
= & \quad , \quad = .
\end{align*}
\]
If the vector space $V$ is $\mathbb{F}^n$, the vectors of $V$ and $V^*$ are ordered lists of $n$ elements of $\mathbb{F}$:

\[
\begin{bmatrix}
-5 \\
8 \\
4 \\
-3 \\
6
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
1, -4, 2, 7, -2
\end{bmatrix}
\quad =
\begin{bmatrix}
-5, 20, -10, -35, 10 \\
8, -32, 16, 56, -16 \\
4, -16, 8, 28, -8 \\
-3, 12, -6, -21, 6 \\
6, -24, 12, 42, -12
\end{bmatrix}.
\]

Here the 5 by 5 matrix is called the Kronecker product of the constituent vectors. The dual pairing (also the trace of the matrix) is then:

\[
\begin{bmatrix}
1, -4, 2, 7, -2
\end{bmatrix}
\quad =
\quad 1(-5) + (-4)(8) + 2(4) + 7(-3) + (-2)(6) \quad = \quad -62.
\]
The sign of a permutation

For \( n \in \mathbb{N} \), the permutation group \( S_n \) is the group of all bijections from \( \mathbb{N}_n \) to itself, with the group multiplication given by composition. Then \( S_n \) has \( n! \) elements.

Note that \( S_1 \) has just one element, the identity element.

For \( n \geq 2 \), put \( D(\underline{x}) = \Pi_{i<j}(x_i - x_j) \), for (commutative) variables \( \underline{x} = (x_1, x_2, \ldots, x_n) \).

For \( s \in S_n \), put \( s(D)(\underline{x}) = \Pi_{i<j}(x_{s(i)} - x_{s(j)}) \).

Since \( s \) is a bijection, the factors \( x_{s(i)} - x_{s(j)} \) are all distinct and not identically zero and each one is of the form \( \epsilon_{k,m}(x_k - x_m) \) for some \( k < m \), with \( k \) and \( m \) in \( \mathbb{N}_n \) and \( \epsilon_{k,m} = \pm 1 \).

There are \( \frac{n(n-1)}{2} \) such factors, the same as in the polynomial \( D(\underline{x}) \), so by the pigeonhole principle, every factor of \( D(\underline{x}) \) occurs exactly once in \( s(D)(\underline{x}) \), so we arrive at the formula:

\[
s(D)(\underline{x}) = \epsilon(s)D(\underline{x}), \quad \epsilon(s) = \pm 1.
\]

If also \( t \in S_n \) and we put \( y_i = x_{t(i)} \), for \( 1 \leq i \leq n \), then we get:

\[
s(D)(\underline{y}) = \epsilon(s)D(\underline{y}) = \epsilon(s)t(D)(\underline{x}) = \epsilon(t)\epsilon(s)D(\underline{x})
\]

\[
= \Pi_{i<j}(y_{s(i)} - y_{s(j)}) = \Pi_{i<j}(x_{t(s(i))} - x_{t(s(j))})
\]

\[
= \Pi_{i<j}(x_{t(s(i))} - x_{t(s(j))}) = (t \circ s)(D)(\underline{x}) = \epsilon(t \circ s)D(\underline{x}).
\]

So the polynomial: \( (\epsilon(t)\epsilon(s) - \epsilon(t \circ s))D(\underline{x}) \) is identically zero, so we infer the relation, valid for any permutations \( s \) and \( t \) in \( S_n \):

\[
\epsilon(t)\epsilon(s) = \epsilon(t \circ s).
\]

This shows that the map \( \epsilon : S_n \to \mathbb{Z}_2 \), \( s \to \epsilon(s) \) is a group homomorphism, where \( \mathbb{Z}_2 = \{1, -1\} \) is the multiplicative group with two elements, 1, the identity and \( -1 \), with \( (-1)^2 = 1 \).

Then \( \epsilon(s) \) is called the sign of the permutation \( s \).

The permutation \( s \) is called even if \( \epsilon(s) = 1 \) and odd if \( \epsilon(s) = -1 \).
Next define the auxiliary polynomials, for any $i$ and $j$ with $1 \leq i \neq j \leq n$:

$$E_{ij}(x) = (-1)^{i+j+1}\Pi_{1 \leq k \leq n, k \neq i, k \neq j}((x_i - x_k)(x_j - x_k)),$$

$$F_{ij}(x) = \Pi_{1 \leq k < l \leq n, k \neq i, l \neq i, k \neq j, l \neq j}(x_k - x_l).$$

Note that, by inspection, $E_{ij}(x)$ and $F_{ij}(x)$ are symmetric under the interchange of $i$ and $j$. Also $F_{ij}(x)$ is independent of the variables $x_i$ and $x_j$. Then we have the formula, highlighting the dependence of the polynomial $D(x)$ on the variables $x_i$ and $x_j$, with $i < j$:

$$D(x) = (x_i - x_j)E_{ij}(x)F_{ij}(x).$$

If now $s \in S_n$ satisfies $s(i) = j$, $s(j) = i$ and $s(k) = k$, for all $k$ with $1 \leq k \leq n$, $k \neq i$ and $k \neq j$, then we have:

$$s(D)(x) = (x_{s(i)} - x_{s(j)})H_{ij}(x)M_{ij}(x),$$

$$H_{ij}(x) = (-1)^{i+j+1}\Pi_{1 \leq k \leq n, k \neq i, k \neq j}((x_{s(i)} - x_{s(k)})(x_{s(j)} - x_{s(k)}))$$

$$= (-1)^{i+j+1}\Pi_{1 \leq k \leq n, k \neq i, k \neq j}((x_j - x_k)(x_i - x_k)) = E_{ij}(x),$$

$$M_{ij}(x) = \Pi_{1 \leq k < l \leq n, k \neq i, l \neq i, k \neq j, l \neq j}(x_{s(k)} - x_{s(l)}) = \Pi_{1 \leq k < l \leq n, k \neq i, l \neq i, k \neq j, l \neq j}(x_k - x_l) = F_{ij}(x).$$

So we get:

$$s(D)(x) = (x_{s(i)} - x_{s(j)})H_{ij}(x)M_{ij}(x) = (x_j - x_i)E_{ij}(x)F_{ij}(x) = -D(x).$$

This proves that $\epsilon(s) = -1$.

The mapping $s$ is called the $(i, j)$ transposition and is denoted $(ij)$. So the map $\epsilon : S_n \to \mathbb{Z}_2$ is surjective, when $n \geq 2$ (and not when $n = 1$, since then $S_1$ consists only of the identity element, which is defined to be even $\epsilon(1) = 1$).

For $n \geq 2$, the kernel of $\epsilon$ is called the alternating group $A_n$, the normal subgroup of $S_n$ of all even permutations, a subgroup of index two in $S_n$, so with $2^{-1}n!$ elements. Every element of the symmetric group is a product of transpositions, so the sign of the element is therefore positive or negative according to the evenness or oddness of the number of transpositions in the product, so the parity of the number of such transpositions is an invariant.
Acting on tensors, we can use permutations to rearrange the arguments of the tensor. This is achieved by acting on the tensor with suitable products of the Kronecker delta tensor. For example in the case of $S_3$, its six elements are represented as:

\[
(1) = \begin{array}{c|c}
1 & 2 \\
2 & 3 \\
3 & 1 \\
\end{array}, 
(231) = \begin{array}{c|c}
1 & 3 \\
3 & 2 \\
2 & 1 \\
\end{array}, 
(312) = \begin{array}{c|c}
1 & 2 \\
2 & 1 \\
3 & 3 \\
\end{array}, 
(23) = \begin{array}{c|c}
1 & 2 \\
2 & 3 \\
3 & 1 \\
\end{array}, 
(13) = \begin{array}{c|c}
1 & 3 \\
3 & 1 \\
2 & 2 \\
\end{array}, 
(12) = \begin{array}{c|c}
1 & 1 \\
2 & 2 \\
3 & 3 \\
\end{array}.
\]

Here $(231)$ denotes the cyclic permutation $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ and $(312)$ denotes the cyclic permutation $1 \rightarrow 3 \rightarrow 2 \rightarrow 1$.

In general, given a word $w = f w_1 w_2 \ldots w_p$ in $\mathcal{T}^p(X)$, for some fixed non-negative integer $p$ and given a permutation $\sigma \in S_p$, we define:

\[
\sigma(w) = f w_{\sigma(1)} w_{\sigma(2)} \ldots w_{\sigma(p)}.
\]

If now $\rho$ is another permutation in $S_p$, we have:

\[
\rho(\sigma(w)) = f w_{\rho(\sigma(1))} w_{\rho(\sigma(2))} \ldots w_{\rho(\sigma(p))}
= f w_{(\rho \circ \sigma)(1)} w_{(\rho \circ \sigma)(2)} \ldots w_{(\rho \circ \sigma)(p)}
= (\rho \circ \sigma)(w).
\]

Then the assignment $w \rightarrow \sigma(w)$ for each word extends naturally to give an endomorphism of $\mathcal{T}^p(X)$, still denoted $\sigma$, such that we have $\rho(\sigma(t)) = (\rho \circ \sigma)(t)$, for any permutations $\rho$ and $\sigma$ in $S_p$ and any $t \in \mathcal{T}^p(X)$. So for each $p$, we get a linear representation of the group $S_p$ on the $n^p$ dimensional vector space $\mathcal{T}^p(X)$. 
The symmetric tensor algebra

The symmetric tensor algebra \( S(X) \) of a vector space \( X \) is the algebra obtained from the tensor algebra \( T(X) \), by requiring that all the letters of all words commute, so the multiplication in this algebra is commutative and indeed is isomorphic to a standard commutative algebra of polynomials in \( n \)-variables over the field \( \mathbb{F} \), when \( X \) has dimension \( n \). So there is a surjective algebra homomorphism from \( T(X) \) to \( S(X) \), which maps each word to itself. For clarity the symmetric tensor product is often represented by the notation \( \odot \). The images of the \( n^p \) basis words \( e_{i_1}e_{i_2}\cdots e_{i_p} \) for \( T^p(X) \), where \( e_i, i = 1, 2, \ldots, n \) is a basis for \( X \), gives the following basis words for the image \( S^p(X) \) of \( T^p(X) \) in \( S(X) \): \( e_{i_1} \odot e_{i_2} \odot \cdots \odot e_{i_p} \), for \( 1 \leq i_1 \leq i_2 \cdots \leq i_p \leq n \).

This basis has \( \binom{n+p-1}{p} \) elements.

There is a natural subspace of \( T(X) \) that maps isomorphically to the symmetric algebra: it is the subspace spanned by all words of the form \( fw^p \), for \( w \in X \) and \( f \in \mathbb{F} \). Explicitly there is an \( \mathbb{F} \)-linear idempotent symmetrization map \( S : T(X) \to T(X) \), which preserves tensor type and which for a given positive integer \( p \), we describe in terms of the action of the symmetric group \( S_p \) on \( T^p(X) \) given above: For each tensor \( t \in T^p(X) \), we define:

\[
S(t) = \frac{1}{p!} \sum_{\sigma \in S_p} \sigma(t).
\]

This gives an endomorphism of \( T^p(X) \). We prove that it is idempotent:

\[
S^2(t) = S \left( \frac{1}{p!} \sum_{\sigma \in S_p} \sigma(t) \right) = \frac{1}{p!} \sum_{\rho \in S_p} \sum_{\sigma \in S_p} \rho(\sigma(t)) = \frac{1}{p!} \sum_{\rho \in S_p} \sum_{\sigma \in S_p} (\rho \circ \sigma)(t)
\]

\[
= \frac{1}{p!} \sum_{\rho \in S_p} \sum_{\sigma \in S_p} (\rho \circ (\rho^{-1} \circ \sigma))(t) = \frac{1}{p!} \sum_{\rho \in S_p} \sum_{\sigma \in S_p} \sigma(t) = \sum_{\sigma \in S_p} \sigma(t) = S(t).
\]

Here we used the fact that as \( \sigma \) runs once through all permutations in \( S_p \), so does \( \rho^{-1} \circ \sigma \), for each fixed permutation \( \rho \) in \( S_p \). The image of \( S \) is then called the subspace of symmetric tensors. On this space we can define the (associative) symmetric tensor product by the formula \( A \odot B = S(AB) \), for any symmetric tensors \( A \) and \( B \). This product makes the space of symmetric tensors into an algebra naturally isomorphic to the algebra \( S(X) \).
For example, given a tensor $T$ in $T^3(\mathbb{X})$, we can represent its symmetrization $S(T)$ diagrammatically as follows:

\[
6S(T) = T + T + T + T + T + T.
\]

Just like any algebra of polynomials, the algebra $S(\mathbb{X})$ carries a natural collection of derivations (or first order derivative operators). A derivation $D$ of $S(\mathbb{X})$ is an endomorphism of $S(\mathbb{X})$, obeying the rules:

- $Dc = 0$, for any $c \in \mathbb{F} = S^0(\mathbb{X})$,
- $D(f + g) = Df + Dg$, for any $f$ and $g$ in $S(\mathbb{X})$,
- $D(fg) = (Df)g + (Dg)f$, for any $f$ and $g$ in $S(\mathbb{X})$.

Then $D$ is determined by its action on $S^1(\mathbb{X}) = \mathbb{X}$, which is an arbitrary linear map from $\mathbb{X}$ to $S(\mathbb{X})$, so $D$ is determined by an element of $S(\mathbb{X}) \otimes \mathbb{X}^*$. If $D$ and $E$ are derivations, so is their bracket $[D, E] = DE - ED$ and this bracket obeys all the usual rules of a Lie bracket (see the section on the Lie derivative along a vector field). Indeed we have a natural linear derivative operator denoted $\partial$, which takes $S(\mathbb{X})$ to $\mathbb{X} \otimes S(\mathbb{X})$ and is determined uniquely by the derivative formula, valid for any $t \in \mathbb{X}$ and any positive integer $p$:

\[
\partial \otimes \frac{t^p}{p!} = t \otimes \frac{t^{p-1}}{(p-1)!}.
\]

Alternatively the operator $\partial$ is determined by the formal power series formula (obtained by summing the previous formula over $p$):

\[
\partial \otimes e^t = t \otimes e^t.
\]
Equivalently, we have, acting on the word $w = f w_1 w_2 \ldots w_p$, with $f \in F$ and each $w_i \in X$, the derivative formula:

$$\partial \otimes (f w_1 w_2 w_3 \ldots w_p) = f w_1 \otimes w_2 w_3 \ldots w_p + f w_2 \otimes w_1 w_3 \ldots w_p + \cdots + f w_p \otimes w_1 w_2 w_3 \ldots w_{p-1}.$$ 

Then the general derivation is given by $D \partial$, where $D$ lies in $S(X) \otimes X^*$. Finally, we can compute multiple derivatives, which (since they commute with each other) give natural maps from $S(X)$ to $S(X) \otimes S(X)$; so for example, we have (when $p \geq k$) and for any $t \in X$:

$$\partial^k \otimes \frac{t^p}{p!} = t^k \otimes \frac{t^{p-k}}{(p-k)!},$$

$$\partial^k \otimes e^t = t^k \otimes e^t,$$

$$e^\partial \otimes e^t = e^t \otimes e^t.$$

The skew-symmetric tensor algebra: the Grassmann or exterior algebra

The skew-symmetric tensor algebra $\Omega(X)$ of a vector space $X$, also called its Grassmann algebra, or its exterior algebra, is the algebra obtained from the tensor algebra, $T(X)$, by requiring that all the individual letters of all words anti-commute, so the multiplication in this algebra is graded commutative: an element is even if it is a sum of words each with an even number of letters and is odd if it is a sum of words each with an odd number of letters. Then even elements commute with everything, whereas odd elements commute with even and anti-commute with each other. In particular, the square of any odd element is zero. There is then a surjective algebra homomorphism from $T(X)$ to $\Omega(X)$, which maps each word to itself. For clarity, the skew-symmetric product is often represented by the notation $\wedge$. The image of the $n^p$ basis words $e_1 e_2 \ldots e_p$ for $T^p(X)$, where $\{e_i, i = 1, 2, \ldots, n\}$ is a basis for $X$, gives the following basis words for the image $\Omega^p(X)$ of $T^p(X)$ in $\Omega(X)$: $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p}$, for $1 \leq i_1 < i_2 \cdots < i_p \leq n$. This basis has $\binom{n}{p}$ elements. In particular the Grassmann algebra of $X$ is finite dimensional, of total dimension $\sum_{p=0}^{n} \binom{n}{p} = 2^n$. 

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There is a natural subspace of $T(X)$ that maps isomorphically to the Grassmann algebra. Explicitly there is an $F$-linear idempotent skew-symmetrization map $\Omega : T(X) \to T(X)$, which preserves tensor type and which for a given positive integer $p$, we describe in terms of the action of the symmetric group $S_p$ on $T^p(X)$ given above. For each tensor $t \in T^p(X)$, we define:

$$\Omega(t) = \frac{1}{p!} \sum_{\sigma \in S_p} \epsilon(\sigma)\sigma(t).$$

Here $\epsilon(\sigma)$ is the sign of the permutation $\sigma$. This gives an endomorphism of $T^p(X)$. We prove that the endomorphism $\Omega$ idempotent: for any fixed positive integer $p$ and any $t \in T^p(X)$, we have:

$$\Omega^2(t) = \Omega \left( \frac{1}{p!} \sum_{\sigma \in S_p} \epsilon(\sigma)\sigma(t) \right) = \frac{1}{p!} \sum_{\rho \in S_p} \sum_{\sigma \in S_p} \epsilon(\rho) \epsilon(\sigma) \rho(\sigma(t))$$

$$= \frac{1}{p!} \sum_{\rho \in S_p} \sum_{\sigma \in S_p} \epsilon(\rho \circ \sigma)(\rho \circ \sigma)(t) = \frac{1}{p!} \sum_{\rho \in S_p} \sum_{\sigma \in S_p} \epsilon(\rho \circ \rho^{-1} \circ \sigma)(\rho \circ (\rho^{-1} \circ \sigma))(t)$$

$$= \frac{1}{p!} \sum_{\rho \in S_p} \sum_{\sigma \in S_p} \epsilon(\sigma)\sigma(t) = \sum_{\sigma \in S_p} \sigma(t) = \Omega(t).$$

Here we used the fact $\epsilon$ is a group homomorphism and that as $\sigma$ runs once through all permutations in $S_p$, so does $\rho^{-1} \circ \sigma$, for each fixed permutation $\rho$ in $S_p$. The image of $\Omega$ is then called the subspace of skew-symmetric tensors. On this space we can define the skew-symmetric tensor product by the formula $A \wedge B = \Omega(AB)$, for any skew-symmetric tensors $A$ and $B$. This product makes the space of skew-symmetric tensors into an algebra naturally isomorphic to the algebra $\Omega(X)$.
For example, given a tensor $T$ in $T^3(X)$, we can represent its skew-symmetrization $\Omega(T)$ diagrammatically as follows:

\[
\begin{align*}
6\Omega(T) &= \ + \ + \ + \ - \ - \ - \\
\end{align*}
\]

Just like the symmetric algebra, the Grassmann algebra $\Omega(X)$ carries a natural derivative operator. A derivation $D$ of $\Omega(X)$ is an endomorphism of $\Omega(X)$, obeying the rules:

- $Dc = 0$, for any $c \in F = \Omega^0(X)$,
- $D(f + g) = Df + Dg$, for any $f$ and $g$ in $\Omega(X)$,
- $D(fg) = (Df)g + (-1)^{pq}(Dg)f$, for any $f \in \Omega^p(X)$ and $g \in \Omega^q(X)$ and any non-negative integers $p$ and $q$.

Then $D$ is determined by its action on $\Omega^1(X) = X$, which is an arbitrary linear map from $X$ to $\Omega(X)$, so $D$ is determined by an element of $\Omega \otimes X^*$. A derivation $D$ is said to have integral degree $k$ if it maps $\Omega^p(X)$ to $\Omega^{p+k}(X)$, for each non-negative integer $p$. If $D$ is a derivation of degree $p$ and $E$ a derivation of degree $q$, so is their bracket $[D, E] = DE - (-1)^{pq}ED$ and this bracket obeys all the usual rules of a graded Lie bracket (see the section on differential forms). Indeed we have a natural linear derivation of degree minus one, denoted $\delta$, which takes $\Omega(X)$ to $X \otimes \Omega(X)$ and is determined uniquely by the derivative formula acting on any word $w = f w_1 w_2 \ldots w_k$, with $f \in F$ and each $w_i \in X$, the derivative formula:

\[
\delta(f w_1 w_2 w_3 \ldots w_k) = f w_1 \otimes w_2 w_3 \ldots w_k - f w_2 \otimes w_1 w_3 \ldots w_k + f w_3 \otimes w_1 w_2 \ldots w_k + \cdots + (-1)^{k-1} f w_k \otimes w_1 w_2 w_3 \ldots w_{k-1}.
\]

Then the general derivation is given by $D, \delta$, where $D$ lies in $\Omega(X) \otimes X^*$. Finally, we can compute multiple derivatives, which (since they anti-commute with each other) give natural maps from $\Omega(X)$ to $\Omega(X) \otimes \Omega(X)$. In particular if $X$ has dimension $n$ all derivatives of order $n + 1$ or more vanish identically.
Determinants

Consider a vector space \(X\) with dual space \(X^*\), defined over a field \(F\), with dimension \(n\), a positive integer. Then \(X\) acts on \(\Omega(X^*)\) the exterior algebra of \(X^*\) by the derivation \(\delta_v\), defined for each \(v \in X\). The operator \(\delta_v\) is the derivation of degree minus one, which kills \(\Omega(\delta_v) = F\) and acting on \(\Omega^1(X^*) = X^*\) obeys the relation, valid for any \(v \in X\) and \(\alpha \in X^*:\)

\[
\delta_v(\alpha) = \alpha(v).
\]

Consider now the vector space \(L = \Omega^n(X^*)\), a vector space over \(F\) of dimension 1, so isomorphic to \(F\). Given \(n\)-vectors, \(v_1, v_2, \ldots, v_n\) in \(X\) and \(\epsilon \in L\), the following quantity is a well-defined element of \(F\):

\[
D(v_1, v_2, \ldots, v_n)(\epsilon) = \delta_{v_1} \delta_{v_{n-1}} \ldots \delta_{v_2} \delta_{v_1} \epsilon.
\]

This quantity is linear in all \((n+1)\)-variables \((v_1, v_2, \ldots, v_n, \epsilon)\). In particular, the quantity \(D(v_1, v_2, \ldots, v_n)\) gives an element of \(L^{-1}\) the dual space of \(L\) (so \(L^{-1}\) is also a vector space of dimension one). Then, from its definition, \(D(v_1, v_2, \ldots, v_n)\) is linear in each argument and totally skew, since the derivations \(\delta_v\), for \(v \in X\) pairwise anti-commute and each is linear in \(v\).

Let \(\{e = e^1\}\) be a basis of \(L\) (so \(e\) is any non-zero element of \(L\)), with dual basis \(e^{-1}\) of \(L^{-1}\), so we have \(e^{-1}(e) = 1\). Then there is a basis \(\{e_1, e_2, \ldots, e_n\}\) for \(X\), with dual basis \(\{e^1, e^2, \ldots, e^n\}\), a basis of \(X^*\), such that \(e = e^1 e^2 \ldots e^n\). Then we have:

\[
D(e_1, e_2, \ldots, e_n)(e) = \delta_{e_1} \delta_{e_{n-1}} \ldots \delta_{e_2} \delta_{e_1}(e^1 e^2 \ldots e^n) = 1 = e^{-1}(e).
\]

So we have the formula \(D(e_1, e_2, \ldots, e_n) = e^{-1}\).

This, in turn, yields the general formula:

\[
D(v_1, v_2, \ldots, v_n) = \det_e(v_1, v_2, \ldots, v_n) e^{-1} = \det_e(v_1, v_2, \ldots, v_n) D(e_1, e_2, \ldots, e_n).
\]

Here the quantity \(\det_e(v_1, v_2, \ldots, v_n)\) is an element of \(F\), and is linear in each vector argument, \(v_1, v_2, \ldots, v_n\) and totally skew. Then \(\det_e\) is called the determinant function, relative to the basis \(\{e\}\) of \(L\). Under the replacement \(e \to f\), where \(\{f\}\) is also a basis for \(L\) we have \(\det_f = r \det_e\), where \(f = re\). The argument above shows that when the vectors \(\{v_i, i = 1, 2, \ldots, n\}\) are linearly independent, so constitute a basis, then \(D(v_1, v_2, \ldots, v_n)\) is non-zero. Conversely, it is easy to see that if the vectors \(v_i, i = 1, 2, \ldots, n\) are linearly dependent, then \(D(v_1, v_2, \ldots, v_n)\) is zero. So we have that \(\det_e(v_1, v_2, \ldots, v_n)\) vanishes if and only if the vectors \(\{v_i, i = 1, 2, \ldots, n\}\) are linearly dependent.
Consider now \( n \)-elements \( \alpha_1, \alpha_2, \ldots, \alpha_n \) of the space \( X^* \). Then we have:

\[
\alpha_1 \alpha_2 \ldots \alpha_n = E(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{L}.
\]

Here the quantity \( E(\alpha_1, \alpha_2, \ldots, \alpha_n) \) is linear in each argument and totally skew. The arguments given above, show that \( E(\alpha_1, \alpha_2, \ldots, \alpha_n) \) vanishes if and only if the \( \{\alpha_i, i = 1, 2, \ldots, n\} \) are linearly dependent. Further we may write \( E(\alpha_1, \alpha_2, \ldots, \alpha_n) = \det e_{-1}(\alpha_1, \alpha_2, \ldots, \alpha_n) \), where \( \det e_{-1}(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{F} \) is linear in each argument and totally skew.

Now consider the element of \( \mathbb{F} \):

\[
D(v_1, v_2, \ldots, v_n) E(\alpha_1, \alpha_2, \ldots, \alpha_n) = \det e(v_1, v_2, \ldots, v_n) \det e_{-1}(\alpha_1, \alpha_2, \ldots, \alpha_n)
\]

\[
= \delta_{v_n} \delta_{v_{n-1}} \ldots \delta_{v_2} \delta_{v_1} (\alpha_1 \alpha_2 \ldots \alpha_n) = \Sigma_{\sigma \in S_n} \epsilon(\sigma) \left( \Pi_{i=1}^n (\alpha_{\sigma(i)})(v_i) \right).
\]

Note that this relation is symmetrical between the vector space \( X \) and its dual \( X^* \):

\[
D(v_1, v_2, \ldots, v_n) E(\alpha_1, \alpha_2, \ldots, \alpha_n) = \delta_{v_n} \delta_{v_{n-1}} \ldots \delta_{v_2} \delta_{v_1} (\alpha_1 \alpha_2 \ldots \alpha_n)
\]

\[
= \Sigma_{\sigma \in S_n} \epsilon(\sigma) \left( \Pi_{i=1}^n v_i (\alpha_{\sigma(i)}) \right) = \Sigma_{\tau \in S_n} \epsilon(\tau) \left( \Pi_{i=1}^n v_{\tau(j)} \right) = \det(\alpha_i(v_j)).
\]

Also this relation embodies the fact that the one-dimensional vector spaces \( \Omega^n(X) \) and \( \Omega^n(X^*) \) are naturally dual.
Volume

In a Euclidean space, of finite dimension, denote the inner product of vectors $v$ and $w$ by $v.w$.

If we represent the elements of the space by column vectors, we can also represent this inner product by $w.v = v^T w = v.w$.

Then the length $L_v$ of a vector $v$ is $L_v = |v| = \sqrt{v.v} = \sqrt{v^T v}$.

Here $T$ applied to a matrix denotes its transpose.

Then $L_v$ is well-defined and is zero if and only if $v = 0$.

The Cauchy-Schwarz inequality is the inequality $|v.w| \leq |v||w|$, with equality if and only if the vectors $v$ and $w$ are linearly dependent.

The real angle $\theta$, with $0 \leq \theta \leq \pi$, between non-zero vectors $v$ and $w$ is given by the formula: $|v||w| \cos(\theta) = v.w$. The Cauchy-Schwarz inequality implies that $\theta$ is well-defined. We say that (possibly zero) vectors $v$ and $w$ are orthogonal or perpendicular, if $v.w = 0$.

The orthogonal group is the group of linear transformations $v \rightarrow L(v)$, preserving length, so for any vector $v$ we have:

$$|L(v)| = |v|, \quad v^T v = v^T L^T L v.$$  

This gives the defining relation for elements of the orthogonal group:

$$LL^T = L^T L = I.$$  

Here $I$ denotes the identity transformation.

A reflection is a linear transformation of the form:

$$v \rightarrow T_n(v) = v - 2(v.n)n, \quad |n| = 1,$$

$$T_n = I - 2n \otimes n^T, \quad n^T n = 1.$$  

If $v$ is any vector we can write $v$ uniquely in the form $v = a + b$, where $a$ is perpendicular to the unit vector $n$ and $b$ is parallel to $n$.

Explicitly we have $b = (v.n)n$ and $a = v - (v.n)n$.

Then the reflection $T_n$ acts so that $T_n(v) = a - b$.

There are then two basic facts about the orthogonal group:

- The group is generated by reflections.

- The parity (even/odd) of the number of reflections whose product is a given element of the orthogonal group is independent of the choice of the reflections producing that element.
More precisely, from the relation $AA^T = I$, we infer that $\det(A)^2 = 1$, so $\det(A) = \pm 1$. Then every reflection $T_n$ has $\det(T_n) = -1$ and an orthogonal transformation $A$ is the product of an even number of reflections if and only if $\det(A) = 1$ and is otherwise the product of an odd number of reflections. The orthogonal transformations of unit determinant form a subgroup of the orthogonal group, called the rotation group.

If now $v$ and $w$ are vectors, they define a parallelogram in the Euclidean space. It has area $A_{v,w}$ given by the formula:

$$A_{v,w} = \sqrt{(v.v)(w.w) - (v.w)^2}.$$  

The Cauchy-Schwarz inequality gives that $A_{v,w}$ is well-defined and $A_{v,w} = 0$ if and only if $v$ and $w$ are linearly dependent, precisely the case where the parallelogram collapses onto a line. More generally, if $\mathcal{P} = \{v_1, v_2, \ldots v_k\}$ is a collection of vectors of the space, then the volume $V_{\mathcal{P}}$ of the parallelepiped determined by them is:

$$V_{\mathcal{P}} = \sqrt{\det(v_i,v_j)}.$$  

One can prove that $V_{\mathcal{P}}$ is well-defined and zero if and only if the vectors of $\mathcal{P}$ are linearly dependent.

We can remove the dependence on the standard basis of Euclidean space, by passing to a real vector space $V$ with a symmetric bilinear form $g : V \times V \to \mathbb{R}$, that is positive definite: for any $0 \neq v \in V$, we have $g(v,v) > 0$. If now $\mathcal{P} = \{v_1, v_2, \ldots v_k\}$ is a collection of vectors of the space $V$, then the volume $V_{\mathcal{P}}$ of the parallelepiped determined by them is:

$$V_{\mathcal{P}} = \sqrt{\det(g(v_i,v_j))}.$$  

Again $V_{\mathcal{P}}$ is well-defined and vanishes if and only if the elements of $\mathcal{P}$ are linearly dependent.
The metric of $V$ induces a metric still called $g$ on the tensor algebra of $V$, such that if $v_1v_2...v_k$ and $w_1w_2...w_m$ are words of the algebra, then their inner product is zero, unless $k = m$ and when $k = m$ we have:

$$g(v_1v_2...v_k, w_1w_2...w_k) = \prod_{j=1}^{k} g(v_i, w_i).$$

This in turn induces inner products on the symmetric tensor algebra, such that in the case of the symmetric algebra we have:

$$g(e^v, e^w) = e^{g(v,w)}.$$

Equivalently, we have, for each nonnegative integer $k$:

$$g(v^k, w^k) = k! g(v, w)^k.$$

Then for example, we have:

$$g(v_1v_2, w_1w_2) = \frac{1}{16} g((v_1 + v_2)^2 - (v_1 - v_2)^2, (w_1 + w_2)^2 - (w_1 - w_2)^2)$$

$$= \frac{1}{8} (g(v_1 + v_2, w_1 + w_2)^2 - g(v_1 + v_2, w_1 - w_2)^2 - g(v_1 - v_2, w_1 + w_2)^2 + g(v_1 - v_2, w_1 - w_2)^2)$$

$$= \frac{1}{8} (g(v_1 + v_2, w_1 + w_2) - g(v_1 + v_2, w_1 - w_2)) (g(v_1 + v_2, w_1 + w_2) + g(v_1 + v_2, w_1 - w_2))$$

$$- \frac{1}{8} ((g(v_1 - v_2, w_1 + w_2) - g(v_1 - v_2, w_1 - w_2)) (g(v_1 - v_2, w_1 + w_2) + g(v_1 - v_2, w_1 - w_2))$$

$$= \frac{1}{2} (g(v_1 + v_2, w_1) g(v_1 + v_2, w_2) - g(v_1 - v_2, w_1) g(v_1 - v_2, w_2))$$

$$= g(v_1, w_1) g(v_2, w_2) + g(v_2, w_1) g(v_1, w_2).$$

The right-hand-side is called a permanent, a variation of the formula for the determinant, where all signs in the formula are taken to be positive. In general, we could use the Laplace or Fourier transform to compute the inner products and hence permanents.

In the case of the exterior algebra, we have:

$$g(v_1v_2...v_k, w_1w_2...w_k) = \det(g(v_i, w_j)).$$

Note in particular that $g(v_1v_2...v_k, v_1v_2...v_k) = \det(g(v_i, v_j)) = \mathcal{V}_P^2$. Here $\mathcal{V}_P$ is the volume of the parallelepiped determined by the vectors $v_1, v_2, ..., v_k$. 39
Determinants and linear transformations

Let $\mathbb{X}$ and $\mathbb{Y}$ be vector spaces (over the same field $\mathbb{F}$) of the same dimension $n$. Put $\omega_\mathbb{X} = \Omega^n(\mathbb{X})$ and $\omega_\mathbb{Y} = \Omega^n(\mathbb{Y})$. These are each one-dimensional vector spaces. Let $L : \mathbb{X} \to \mathbb{Y}$ be a linear transformation. Then $L$ induces a natural linear mapping called $T(L)$ mapping the tensor algebra of $\mathbb{X}$ to that of $\mathbb{Y}$, such that if $x = f x_1 x_2 \ldots x_k$ is a word of $T(\mathbb{X})$, then we have $T(L)(x) = f L(x_1) L(x_2) \ldots L(x_k)$. This mapping is compatible with the passage to the exterior algebra, so we induce a natural map called $\Omega(L)$ mapping $\Omega(\mathbb{X})$ to $\Omega(\mathbb{Y})$. This mapping preserves degree, so, in particular, in the highest dimension, it induces a natural map, denoted $\det(L)$ from the line $\omega_\mathbb{X}$ to the line $\omega_\mathbb{Y}$. Clearly this map is functorial: if $M : Y \to Z$ is also a linear map with $Z$ a vector space of dimension $n$ over $\mathbb{F}$, then we get $\det(M \circ L) = \det(M) \circ \det(L)$. Also we have that $\det(cL) = c^n \det(L)$, for any $c \in \mathbb{F}$ and $\det(L) \neq 0$, if and only if $L$ is an isomorphism. If $K$ and $L$ are linear maps from $\mathbb{X}$ to $\mathbb{Y}$, their relative characteristic polynomial is the polynomial $\det(sK + tL)$ and is a homogeneous polynomial of total degree $n$, of the form $s^n \det(K) + \cdots + t^n \det(L)$. Of interest also is the restriction of $\Omega(L)$ to the $n$-dimensional space $\Omega^{n-1}(\mathbb{X})$. This is called the adjoint mapping of $L$, and is denoted $\text{adj}(L) : \Omega^{n-1}(\mathbb{X}) \to \Omega^{n-1}(\mathbb{Y})$. Note that we have the formulas:

$$\mathbb{X} \wedge \Omega^{n-1}(\mathbb{X}) = \Omega^n(\mathbb{X}),$$

$$L(w) \wedge \text{adj}(L)(\alpha) = \det(L)(w \wedge \alpha),$$

for any $w \in \mathbb{X}$ and any $\alpha \in \Omega^{n-1}(\mathbb{X})$.

In the special case that $\mathbb{Y} = \mathbb{X}$, so $L$ is an endomorphism of $\mathbb{X}$, then $\det(L)$ is a multiple of the identity isomorphism of $\omega_\mathbb{X}$, by an element of $\mathbb{F}$, so may be regarded canonically as an element of $\mathbb{F}$. Then $\det$ gives a homomorphism from the space of endomorphisms of $\mathbb{X}$ under composition to the field $\mathbb{F}$ under multiplication, whose restriction to the group of isomorphisms of $\mathbb{X}$ is a group homomorphism to the group $\mathbb{F}^*$ of non-zero elements of $\mathbb{F}$ under multiplication. In particular we have $\det(\text{id}_\mathbb{X}) = 1$. Also the characteristic polynomial $\det(s \text{id}_\mathbb{X} + tL)$ is a homogeneous polynomial in the variables $s$ and $t$ of total degree $n$, of the form $s^n + \text{tr}(L)s^{n-1}t + \cdots + t^n \det(L)$. 

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