Vector analysis Quiz 1 Solutions, 5/13/10

Question 1

On a building site a horizontal girder $CD$ of length 60 feet and of weight 4000 pounds is suspended at its end points $C$ and $D$ by two wires $CA$ and $DB$ from two points $A$ and $B$ on a horizontal ceiling, where the points $A$ and $B$ are 100 feet apart and $CA = DB$ (the point $C$ being nearer to $A$ than to $B$).

The girder is 15 feet below the ceiling.

If the breaking strain of each wire is 4000 pounds weight and a fifty percent error margin is required, is the suspension safe?

Explain your answer.

We see that the suspension is symmetrical.

Let $E$ be vertically above $C$ on the line $AB$ and $F$ be vertically above $D$ on the line $AB$.

Then $AE = FB$ and $EF = 60$, so $AB = 100 = AE + EF + FB = 2AE + 60$, so $AE = FB = 20$. So the vectors $CA$ and $DB$ are:

$$CA = [-20, 15], \quad DB = [20, 15].$$

Then we have:

$$|CA| = |DB| = \sqrt{20^2 + 15^2} = \sqrt{400 + 225} = \sqrt{625} = 25.$$  

Then the force in the left wire holding up the girder is $T = tCA$ and the force in the wire $DB$ holding up the girder is $U = DB$, for some scalars $t$ and $u$. The force balance equation for the girder is then:

$$T + U + [0, -4000] = [0, 0],$$

$$[-20t, 15t] + [20u, 15u] = [0, 4000],$$

$$-20t + 20u = 0, \quad t = u,$$

$$15t + 15u = 4000,$$

$$30t = 4000,$$

$$t = u = \frac{400}{3}.$$
Back substituting, we get:

\[ T = \frac{400}{3} [-20, 15] = \left[ -\frac{8000}{3}, 2000 \right], \]

\[ U = \frac{400}{3} [20, 15] = \left[ \frac{8000}{3}, 2000 \right], \]

\[ |T| = |U| = \frac{400}{3} |CA| = \frac{400}{3} \left( \frac{25}{2} \right) = \frac{10000}{3} = 3333.\overline{3}. \]

if we add a fifty percent safety margin, we get:

\[ \frac{3}{2} |T| = \frac{3}{2} |U| = \frac{10000}{3} \left( \frac{3}{2} \right) = 5000. \]

Since this exceeds the limit of 4000 pounds weight, the suspension is unsafe.

Alternatively, we see by symmetry that the forces in the wires have the same strength \( |T| \) and resolving in the vertical direction we get, writing \( \gamma \) radians for the angle \( ACE \):

\[ 2|T| \cos(\gamma) = 4000, \]

\[ |T| = \frac{2000}{\cos(\gamma)} = 2000 \sec(\gamma). \]

But we have:

\[ \sec(\gamma) = \frac{CA}{CE} = \frac{25}{15} = \frac{5}{3}. \]

So we get:

\[ |T| = 2000 \left( \frac{5}{3} \right) = \frac{10000}{3} = 3333.\overline{3}. \]

Then adding a fifty percent margin gives 5000 pounds weight, so the suspension is unsafe, as before.
Question 2

Let \( A = (3, -1, 2), \ B = (-1, 3, 0) \) and \( C = (4, 1, 4) \) be points in space.  
Prove that the triangle \( ABC \) is right-angled and determine its angles, side lengths and its area.

The vectors corresponding to the sides of the triangle are:

\[
X = BC = C - B = [4, 1, 4] - [-1, 3, 0] = [5, -2, 4], \\
Y = CA = A - C = [3, -1, 2] - [4, 1, 4] = [-1, -2, -2], \\
Z = AB = B - A = [-1, 3, 0] - [3, -1, 2] = [-4, 4, -2].
\]

Check: \( X + Y + Z = [5, -2, 4] + [-1, -2, -2] + [-4, 4, -2] = [0, 0, 0]. \)

Then we have:

\[
|X| = \sqrt{5^2 + (-2)^2 + 4^2} = \sqrt{25 + 4 + 16} = \sqrt{45} = 3\sqrt{5}, \\
|Y| = \sqrt{(-1)^2 + (-2)^2 + (-2)^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3, \\
|Z| = \sqrt{(-4)^2 + 4^2 + (-2)^2} = \sqrt{16 + 16 + 4} = \sqrt{36} = 6.
\]

Then we have:

\[
|X|^2 = 45 = 9 + 36 = |Y|^2 + |Z|^2.
\]

So \( ABC \) is right-angled, with hypotenuse \( X = BC \), so the right-angle is at \( A. \)

Check:

\[
CA.AB = Y.Z = [-1, -2, -2].[-4, 4, -2] = (-1)(-4) + (-2)(4) + (-2)(-2) = 4 - 8 - 4 = 0.
\]

So \( CA \) and \( AB \) are perpendicular, as required and the angle \( \alpha \) at \( A \) is 90 degrees, or \( \frac{\pi}{2} \approx 1.5708 \) radians. Then the angle \( \beta \) at \( B \) is:

\[
\beta = \arctan \left( \frac{|Y|}{|Z|} \right) = \arctan \left( \frac{3}{6} \right) = \arctan \left( \frac{1}{2} \right) = 0.4636 \text{ radians} \approx 26.565 \text{ degrees}.
\]

Finally the angle \( \gamma \) at \( C \) is:

\[
\gamma = \arctan \left( \frac{|Z|}{|Y|} \right) = \arctan \left( \frac{6}{3} \right) = \arctan (2) = 1.1071 \text{ radians} \approx 63.435 \text{ degrees}.
\]
Alternatively, we can use our cosine formulas (note the signs here):

$$
\cos(\beta) = \frac{BC \cdot BA}{|BC||BA|} = -\frac{X \cdot Z}{|X||Z|} = -\frac{5(-4) + (-2)(4) + 4(-2)}{3\sqrt{5}(6)}
$$

$$
= -\left(\frac{-20 - 8 - 8}{18\sqrt{5}}\right) = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5},
$$

$$
\beta = \arccos \left(\frac{\sqrt{5}}{10}\right) = 0.4636 \text{ radians} = 26.565 \text{ degrees},
$$

$$
\cos(\gamma) = \frac{CA \cdot CB}{|CA||CB|} = -\frac{Y \cdot X}{|Y||X|} = -\frac{(-1)(5) + (-2)(-2) + (-2)(4)}{3(3\sqrt{5})}
$$

$$
= -\left(\frac{-5 + 4 - 8}{9\sqrt{5}}\right) = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5},
$$

$$
\gamma = \arccos \left(\frac{\sqrt{5}}{5}\right) = 1.1071 \text{ radians} = 63.435 \text{ degrees}.
$$

The area $\Delta$ of the triangle $ABC$, since it is right-angled at $A$ is:

$$
\Delta = \frac{1}{2}|Y||Z| = \frac{1}{2}(3(6)) = 9.
$$

Alternatively we could use the formula:

$$
4\Delta^2 = |AB|^2 |BC|^2 - (AB \cdot BC)^2 = |Z|^2 |X|^2 - (Z \cdot X)^2
$$

$$
= 36.45 - ([-4, 4, -2],[5, -2, 4])^2
$$

$$
= 1620 - (-20 - 8 - 8)^2
$$

$$
= 1620 - 1296
$$

$$
= 324 = 18^2 = 4(9)^2.
$$

So $\Delta^2 = 9^2$ and $\Delta = 9$, as before.
Question 3

Let \( A = (2, 5), \ B = (5, -1) \) and \( C = (7, 5). \)

Find the point \( D \) on the line \( AB \) closest to \( C. \)
Also find the area of the triangle \( ABC. \)

We have:
\[
AB = B - A = [3, -6],
\]
\[
D = A + tAB = [2, 5] + t[3, -6] = [2 + 3t, 5 - 6t].
\]

Then the vector \( CD \) is
\[
CD = D - C = [2 + 3t, 5 - 6t] - [7, 5] = [3t - 5, -6t].
\]

We want this vector to be perpendicular to \( AB, \) so we want:
\[
0 = AB \cdot CD = [3, -6] \cdot [3t - 5, -6t] = 3(3t - 5) - 6(-6t) = 45t - 15,
\]
\[
45t = 15, \quad t = \frac{15}{45} = \frac{1}{3},
\]
\[
D = [2 + 3t, 5 - 6t] = \left[ 2 + 3 \left( \frac{1}{3} \right), 5 - 6 \left( \frac{1}{3} \right) \right] = [2 + 1, 5 - 2] = [3, 3].
\]

Then we have:
\[
CD = D - C = [3, 3] - [7, 5] = [-4, -2],
\]
\[
|CD| = \sqrt{(-4)^2 + (-2)^2} = \sqrt{16 + 4} = \sqrt{20} = 2\sqrt{5}.
\]

Also we have:
\[
|AB| = ||3, -6|| = \sqrt{3^2 + (-6)^2} = \sqrt{9 + 36} = \sqrt{45} = 3\sqrt{5}.
\]

Then the triangle \( ABC \) has base \( b = |AB| \) and height \( h = |CD|, \) so its area \( \Delta \) is:
\[
\Delta = \frac{bh}{2} = \frac{3\sqrt{5}(2\sqrt{5})}{2} = 3(5) = 15.
\]
Alternative formulas for the area:

\[ AB = [3, -6], \]
\[ AC = C - A = [7, 5] - [2, 5] = [5, 0], \]
\[ 2\Delta = \det \begin{vmatrix} AB \\ AC \end{vmatrix} \]
\[ = \det \begin{vmatrix} 3 & -6 \\ 5 & 0 \end{vmatrix} = 3(0) - (-6)5 = 30, \]
\[ \Delta = 15. \]

Alternatively we have:

\[ AB = [3, -6], \ AC = [5, 0], \]
\[ |AB|^2 = 3^2 + (-6)^2 = 9 + 36 = 45, \]
\[ |AC|^2 = 5^2 + 0^2 = 25, \]
\[ AB \cdot AC = [3, -6] \cdot [5, 0] = 3(5) + (-6)0 = 15, \]
\[ 4\Delta^2 = |AB|^2|AC|^2 - (AB \cdot AC)^2 = 45(25) - 15^2 = 1125 - 225 = 900, \]
\[ \Delta^2 = \frac{900}{4} = 225 = 15^2, \]
\[ \Delta = 15. \]
Alternative approaches to getting $D$: we resolve the vector $Y = AC = C - A = [5, 0]$ into pieces $Y = Y_1 + Y_2$, with $Y_1$ parallel to $AB$ and $Y_2$ perpendicular to the direction $AB = [3, -6]$. Then $D = A + Y_1$.

We have:

$$Y_1 = \frac{(AC \cdot AB) AB}{|AB|^2} = \frac{[5, 0] \cdot [3, -6]}{3^2 + (-6)^2} AB = \frac{15}{45} AB = \frac{1}{3} [3, -6] = [1, -2],$$

$$D = A + Y_1 = [2, 5] + [1, -2] = [3, 3].$$

Alternatively we minimize the square of the distance $L$ from $C$ to the line $AB$. A generic point on the line $AB$ is:

$$X = A + t(B - A) = [2, 5] + t[3, -6] = [2 + 3t, 5 - 6t],$$

$$CX = X - C = [2 + 3t, 5 - 6t] - [7, 5] = [3t - 5, -6t],$$

$$L^2 = |CX|^2 = (3t - 5)^2 + (-6t)^2 = 9t^2 - 30t + 25 + 36t^2 = 45t^2 - 30t + 25,$$

$$\frac{dL^2}{dt} = 90t - 30 = 0,$$

$$t = \frac{30}{90} = \frac{1}{3},$$

$$L^2 = 45 \left( \frac{1}{3} \right)^2 - 30 \left( \frac{1}{3} \right) + 25 = 5 - 10 + 25 = 20,$$

$$L = \sqrt{20} = 2\sqrt{5},$$

$$X = [2 + 3t, 5 - 6t] = [2 + 3 \left( \frac{1}{3} \right), 5 - 6 \left( \frac{1}{3} \right)] = [2 + 1, 5 - 2] = [3, 3].$$

The only critical point must be the absolute minimum, since $L^2$ goes to $\infty$ as $t \to \infty$, so must have at least one minimum and we have only one candidate.
Alternatively, the slope of \( AB \) is \(-2\), so the slope of \( CD \) is \( \frac{1}{2} \).

So the direction vector of \( CD \) may be taken to be \([2, 1]\). So the parametric equation of \( CD \) is \( X = [7, 5] + s[2, 1] \).

The parametric equation of \( AB \) is \( X = [2, 5] + t[3, -6] \).

The two lines meet at the point \( X = D \), where:

\[
[2, 5] + t[3, -6] = [7, 5] + s[2, 1],
\]

\[
3t + 2 - 7 - 2s, -6t + 5 - s - 5 = [0, 0],
\]

\[
3t - 2s - 5 = 0, -6t - s = 0,
\]

\[
s = -6t,
\]

\[
3t - 2(-6t) - 5 = 0,
\]

\[
15t = 5, \quad t = \frac{1}{3}, \quad s = -6t = -2.
\]

\[
D = [7, 5] + (-2)[2, 1] = [7 - 4, 5 - 2] = [3, 3],
\]

\[
D = [2, 5] + \frac{1}{3}[3, -6] = 2 + 1, 5 - 2 = [3, 3].
\]