Complex Variables, Spring 2010
Homework Assignments

Homework 1, due Thursday January 14th

Do the following two problems and the four problems from chapter one of Spiegel: 71, 73, 74, 123 (omit the fourth part and the last two parts of this problem).

Speigel 1.71

Describe and graph the following curves in the complex plane:

- $|z - i| = 2$
- $|z + 2i| + |z - 2i| = 6$
- $|z - 3| - |z + 3| = 4$
- $z(\overline{z} + 2) = 3$
- $\Im(z^2) = 4$

Speigel 1.73

Sketch the following regions in the complex plane and for each region discuss whether or not it is closed, open, bounded, unbounded, compact or non-compact.

- $1 < |z + i| \leq 2$
- $\Re(z^2) > 1$
- $|z + 3i| > 4$
- $|z + 2 - 3i| + |z - 2 + 3i| < 10$

Spiegel 1.74

Show that the Cartesian equation of the ellipse $|z + 3| + |z - 3| = 10$ can be written:

$$\frac{x^2}{25} + \frac{y^2}{16} = 1.$$

Speigel 1.123

Let regions $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ in the complex plane be given as follows:

$\mathbb{A} = \{z \in \mathbb{C} : |z + i| < 3\}$ $\mathbb{B} = \{z \in \mathbb{C} : |z| < 5\}$ $\mathbb{C} = \{z \in \mathbb{C} : |z + 1| < 4\}$

Sketch $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ and the following sets:

- $\mathbb{A} \cap \mathbb{B} \cap \mathbb{C}$
- $\mathbb{A} \cup \mathbb{B} \cup \mathbb{C}$
- $\mathbb{A} \cap (\mathbb{B} \cup \mathbb{C})$
- $(\mathbb{A} \cup \mathbb{B}) \cap (\mathbb{B} \cup \mathbb{C})$

Question 1

Let $a = 1 + 2i$ and $b = 1 + i$.
Sketch the complex numbers a, b, $a + b$, $a - b$, a^2, ab, $a\bar{b}$, b^2, ab^{-1} and ba^{-1}.
For each of these numbers, determine their modulus and argument.

Question 2

A complex number z obeys the relations $|z + 1 + 2i| = \sqrt{5}$ and $|z - 3| = 3$.
What can we say about the number z?
In particular is z unique?
Explain your answer graphically.

Homework 2, due Tuesday January 21st

Question 1

Let ω be a cube root of 1 that is not 1 itself (so $\omega^3 = 1$, but $\omega \neq 1$).
Show that $\omega^2 + \omega + 1 = 0$.
Show that ω^2 is also a cube root of 1 that is not 1 and is not ω.
Also express the quantity $z = \dfrac{\omega + 1}{\omega - 1}$ as a linear combination of ω and 1 with real coefficients.

Question 2

Solve (algebraically) the equations $z^2 = 7 + 24i$ and $z^4 = 7 + 24i$ and plot the solutions on the complex plane.

Question 3

Find all square roots of $5 + 12i$ and $12 - 5i$ and illustrate your results in the complex plane.

Question 4

Prove that in the triangle inequality $|z + w| \leq |z| + |w|$, we have equality if and only if either z or w is zero, or both are non-zero and lie on a straight line through the origin with both points on the same line segment emerging from the origin (i.e. the origin does not lie between the points z and w).

Question 5

Find the equation of the circle that passes through the points i, $2 + 3i$ and $2 + i$.
Also find the equation of the tangent line to the circle, through the point $2 + 3i$.

Question 6

Give the geometric interpretation of each of the following complex inequalities, with a sketch.
Also discuss whether or not the set described by the solutions of the equation is open, closed, compact or non-compact and whether or not the solution set has interior points.

- $(3+i)z - (3-i)\overline{z} \neq 6i$
- $|z - 2i + 1|^2 \leq |z + 1|^2$
- $z\overline{z} - (2+i)z - (2-i)\overline{z} \leq 20.$

Homework 3, due Thursday 4th February 2010

Question 1

Let $f(z) = \dfrac{z - 1 + i}{z + 1 + i}$.
Give the domain of $f(z)$ and write $f(z)$ as $u + iv$, where u and v are functions of the real variables x and y, with $z = x + iy$. Verify that the functions u and v obey the Cauchy-Riemann equations, $u_x = v_y$ and $u_y = -v_x$.

Question 2

The electric field at a point z of the complex plane, due to a charged line carrying charge q, perpendicular to the plane and passing through the plane at point w is $\dfrac{q}{\overline{z} - \overline{w}}$, where $z \neq w$.
The total electric field due to an assemblage of such charged lines is the sum of the individual fields.
One line of charge q passes through the origin, and two others, each carrying charge $4q$ pass through the points $4 + 3i$ and $4 - 3i$.
Where is the total field zero?

Question 3

Solve the polynomial equation $z^4 - 24z^2 + 169 = 0$ and plot the roots on the complex plane. The roots should be written in the form $a + b\sqrt{2}$, where a and b involve only I and rational numbers.

Question 4

Let $w = (1 + i)z + 3 - 4i$, defined for any complex z.
Find the image in the w-plane of the following sets:

- $\{z : \Im(z) > 0\}$
- $\{z : \Re(z) = \Im(z)\}$
- $\{z : |z| = 1\}$.

Question 5

Let $w = \dfrac{1}{z}$, defined for any complex $z \neq 0$.

Find the image of the following sets:

- $\{z : |z| = 4\}$
- $\{z : |z-1| = 1\}$
- $\{z : \Re(z) = 2\}$
- $\{z : \Re(z) + \Im(z) = 0\}$

Question 6

Find the compositions $T_4 = T_2 \circ T_1$ and $T_3 = T_1 \circ T_2$ of the transformations $T_1 : z \to z^{-1}$ and $T_2 : z \to 1 - z$.
Give the domain of each of T_3 and T_4.
Show that T_4 and T_3 are inverses of each other.
Also determine the transformations T_4^2 and T_3^2.

Homework 4, due Tuesday 16th February 2010

Question 1

Compute the following limits, or show that the limit in question does not exist

- $\lim_{z \to 3i+4} \frac{|z-3i-4|^2}{z^2-8z+25}$
- $\lim_{z \to 3i} \frac{z^4-81}{(z-2-3i)(z^2-9iz-18)}$

Question 2

Describe the transformation $T : z \to \frac{1}{z-1}$ and determine the images under the transformation T of the circle center $(4, -4)$ and radius 5 and of the line $x + y = 0$. Also graph the curves and their images on one graph.

Question 3

Consider the function $f(z) = z^{\frac{1}{2}}(8-z)^{\frac{1}{2}}$.

- If its value at $z = 4$ is taken to be 4, with cuts along the positive real axis from 8 to infinity and along the negative real axis from 0 to minus infinity, determine the jump in its value as $z = 9$ is approached from above, vis a vis its value as $z = 9$ is approached from below.
- If its value at $z = -1$ is taken to be $3i$, with cuts along the real axis from 0 to 8 determine its value at $z = 9$.

Question 4

Determine directly, from the definition of the derivative, the derivative of the function $f(z) = 2z(z^2-1)^{-1}$ and verify that $f'(z) = \frac{\partial}{\partial x} f(z) = -i \frac{\partial}{\partial y} f(z)$.

Question 5

Show that $u(x,y) = x^3 - 6x^2y - 3xy^2 + 2y^3 - 2xy$ is harmonic and find its harmonic conjugate $v(x,y)$.
Write $f(x,y) = u(x,y) + iv(x,y)$ as a polynomial in the complex variable $z = x + iy$.

Question 6

Show that $u(x,y) = \dfrac{(x^2 - y^2 + 2xy)}{(x^2 + y^2)^2}$ is harmonic and find its harmonic conjugate $v(x,y)$.
Write $f(x,y) = u(x,y) + iv(x,y)$ as a function of the complex variable $z = x + iy$.

Homework 5, due Thursday 25th February 2010

Question 1

Find, from first principles the complex derivative of the function:

$$f(z) = \frac{2z-1}{z^4+1}.$$

Also give the domain(s) of f and of its derivative.

Question 2

Consider the function $f(z) = \ln(z) - \ln(2-z)$.

- If its value at $z = 1$ is taken to be 0, with cuts along the positive real axis from 2 to infinity and along the negative real axis from 0 to minus infinity, determine the jump in its value as $z = 3$ is approached from above, vis à vis its value as $z = 3$ is approached from below.
- If instead its value at $z = -1$ is taken to be $-\ln(3) - \pi i$, with a cut along the real axis from 0 to 2, determine its value at $z = 3$.
- Suppose that we want the function f to be well-defined and continuous on the circle centered at the origin, with radius 5. What does this tell us about the cuts that are allowed for the function f? Explain your answer.

Question 3

Determine (with proof) the following limits, or explain why the limit in question does not exist:

- $\displaystyle\lim_{z\to(1+i)} \left(\frac{z^4-4}{z^3-z^2-iz^2-z+1+i}\right).$
- $\displaystyle\lim_{z\to i} \left(\frac{z^2-\overline{z}^2}{\overline{z}+i}\right).$

Question 4

Determine, with proof the following quantities:

- All possible values of i^i
- All complex z, for which $\sin(z) = i$.

Question 5

Prove that the function $z^3 + |z|^3$ is nowhere analytic, but has a complex derivative at $z = 0$. Also determine, with proof, its complex derivative there.

Question 6

Prove that the function $u(x, y) = e^x(y\cos(y) + x\sin(y))$ is harmonic and find its harmonic conjugate $v(x, y)$. Also write $f = u(x, y) + iv(x, y)$ as a function of $z = x + iy$ and determine the differential df.

Homework 6, due Thursday 4th March 2010

Question 1

Using L'Hopital's rule appropriately, determine the following limits:

- $\lim_{z \to 0} \left(\frac{1 - \cos(z) + z\sin(z)}{z\sin(z)} \right)$.
- $\lim_{z \to 0} \left((\cos(z))^{\frac{1}{z^2}} \right)$.

Question 2

If a transformation $X = u(x, y)$, $Y = v(x, y)$ is given, its Jacobian $J(u, v)$ is the quantity:

$$J(u, v) = u_x v_y - u_y v_x.$$

The transformation is then locally invertible if and only if $J(u, v)$ is non-zero. Prove that if $f = u + iv$ is an analytic function of $z = x + iy$, then:

$$J(u, v) = |f'(z)|^2.$$

Conclude that the transformation is locally invertible if and only if $f'(z)$ is non-vanishing.

Question 3

Let $\alpha = \overline{z}dz$.
Find the integral of α from the point $A = (0, 0)$, to the point $B = (1, 1)$:

- Along the straight line AB.
- Along the parabola, $y = x^2$.
- Along the straight line AC and then along the line CB, where C is the point $C = (0, 1)$.

Question 4

Let $\beta = \dfrac{dz}{z-1}$.
Evaluate the integral of β:

- Taken over the circle $|z-1| = 1$ traced once around, counter-clockwise.
- Taken over the semi-circle $|z| = 2$, from $z = 2$ to $z = -2$, traced counter-clockwise.
- Taken over the semi-circle $|z| = 2$, from $z = 2$ to $z = -2$, traced clockwise.

Question 5

Let $\gamma = z^2 dz + \overline{z}^2 dz$.
Evaluate the integral of γ:

- Taken over the circle $|z| = 1$, traced once around, counter-clockwise.
- Taken over the circle $|z-1| = 1$, traced once around, counter-clockwise.

Question 6

Show that $u(x, y) = \cos(x)e^y + e^x \sin(y)$ is harmonic and find its harmonic conjugate. Hence solve the differential equation:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 8u(x, y).$$

Homework 7, due Thursday 25th March 2010

Question 1

Evaluate the line integral of the differential $\alpha = z^2|z|^2 dz$, taken over the square with vertices $(1,1), (-1,1), (-1,-1), (1,-1)$, taken counter-clockwise.

Question 2

Let $\beta = \dfrac{dz}{2-z}$. Find the integral of β taken around the following contours, each taken once around counter-clockwise:

- The circle $|z-2| = 1$.
- The circle $|z-i| = 1$.
- The circle $|z| = 10^{10}$.

Question 3

Evaluate the integral of the differential $\gamma = (z^2+1)^{-1}dz$ over each of the following contours, each taken counter-clockwise:

- The circle of radius $\dfrac{1}{2}$ center the origin.
- The circle of radius 2, center the origin.
- The ellipse with equation $x^2 + \dfrac{(y-2)^2}{4} = 1$.

Question 4

Let $\mathcal{R}$ be a region with boundary $\partial\mathcal{R}$. Prove that:

$$\int_{\partial\mathcal{R}} \overline{z}dz = -\int_{\partial\mathcal{R}} zd\overline{z} = 2i\mathcal{A}.$$

Here $\mathcal{A}$ is the area of the region $\mathcal{R}$.

Question 5

Let $\alpha = \overline{z}dz$.
Evaluate the integral of α taken over the top half of the ellipse with equation $x^2 + 4y^2 = 4$, taken counter-clockwise with respect to the origin.

Question 6

Let $\alpha = ze^z dz + \overline{z}e^{\overline{z}} d\overline{z}$. Find the integral of α, taken along the parabola $y = x^2$, from the point $A = (2, 4)$ to the point $B = (4, 16)$.

Homework 8, due Thursday 1st April 2010

Question 1

Compute the following integral:

$$\int_{\mathcal{B}} \frac{z^2}{(z^2+1)^2} dz.$$

Here $\mathcal{B}$ is the square with vertices $\pm 2 \pm 2i$, taken once counter-clockwise.

Question 2

Compute the following integral:

$$\int_{\mathcal{C}} \frac{1}{z^3+1} dz.$$

Here the contour $\mathcal{C}$ is the circle center $(1, 1)$ and radius 2, traced once counter-clockwise.
Begin by sketching $\mathcal{C}$ and the roots of $z^3 + 1$ on one graph.

Question 3

Compute the following integral:

$$\int_{\mathcal{D}} \frac{1}{(z^2-4z+5)^2} dz.$$

Here the contour $\mathcal{D}$ is the circle center $(2, 2)$ and radius 2, traced once counter-clockwise.
Begin by sketching $\mathcal{D}$ and the roots of $z^2 - 4z + 5$ on one graph.

Question 4

For n any integer, define the integral $\mathcal{J}_n$ by the formula:

$$\mathcal{J}_n = \int_{\mathcal{E}} z^n \sin(z) dz.$$

Here $\mathcal{E}$ is the unit circle, center the origin, traced once counter-clockwise.
Determine the integral $\mathcal{J}_n$.

Question 5

Let $f(z)$ be analytic in the plane and suppose that $|f(z)| < |z|$, whenever $|z| \geq R$, for some positive real number R.
Prove that $f(z)$ is linear: $f(z) = az + b$ for some complex constants a and b, where $|a| < 1$.

Question 6

Show that the average value of a harmonic function, taken over a circle, equals the value of the function at the center of the circle.

Homework 9, due Thursday 15th April 2010

Question 1

Let $\alpha = \dfrac{(z^2 + 2z - 5)dz}{(z^2 + 4)(z^2 + 2z + 2)}$.
Find the integral of α taken over the circle center the origin and radius 10^{10}, traced once around counter-clockwise. If Γ is a circular contour which goes once around counter-clockwise avoiding the singularities of α, how many different numbers can be obtained as integrals $\int_\Gamma \alpha$? Explain your answer.

Question 2

Let Γ denote the circle center the origin and radius one traced once around counter-clockwise.
Let the function $f(z) = z^{\frac{1}{2}}$ be defined such that there is a cut from the origin to infinity along the negative real axis and such that $f(1) = 1$.
Compute $\int_\Gamma f(z)dz$.
Suppose instead that $f(1) = 1$ and the cut goes from 0 to infinity along the positive imaginary axis.
Does $\int_\Gamma f(z)dz$ have the same value? Explain your answer.

Question 3

Let $\beta = \dfrac{dz}{z^2 + 9}$.
Evaluate the integral of β taken over:

- The parabola $y = x^2$, taken over all x-values, in the direction of increasing x.
- The line $y = x - 2$, taken over all x-values, in the direction of increasing x.

Question 4

Let $\gamma = \dfrac{(e^{-\pi z} + \cos(\pi z))dz}{(z^2 + 4)^2}$.
Find the integral of γ taken over the circle center the origin and radius 10^{10}, traced once around counter-clockwise.

Question 5

Let $\delta = \dfrac{dz}{\tan^2(z)}$.
Find the integral of δ taken over the circle center the origin and radius 100, traced once around counter-clockwise.

Question 6

Prove that all the roots of $z^5 + z - 16i$ lie in the annular region $1 < |z| < 2$.

Homework 10, due Thursday 22nd April 2010

Question 1

- Let $A = \sum_{n=0}^{\infty} \frac{e^{2\pi i n z}}{(n^2+1)^{\frac{3}{2}}}$.

  Prove that A converges if and only if $\Im(z) \geq 0$.

- Let $B = 1 + 2z + z^2 + 2z^3 + z^4 + 2z^5 + \ldots$.
  Prove that B converges if and only if $|z| < 1$ and find its sum in that case.

Question 2

Find the Taylor series, based at the origin, for the function: $f(z) = \frac{1}{(z+2)(z^2+2z+5)}$.
When does this series converge? Explain your answer.

Question 3

Let $g(z) = \frac{3z-3}{(2z-1)(z-2)}$.

Find Laurent series for the function $g(z)$ centered at $z = 1$, for the annuli:

- $\frac{7}{6} < |z| < \frac{4}{3}$.
- $3 < |z| < 10^{10}$.

Question 4

By using appropriate series expansions (*not* using L'Hôpital), compute the following limits:

- $\lim_{z \to 0} \left(\frac{e^{2z} - 2\sin(z) - \cos(2z)}{\tan^3(z)} \right)$
- $\lim_{z \to 1} \left(\frac{(3+z)^{\frac{3}{2}} + 9\ln(z) - (1+z)^3}{(z-1)^2} \right)$

Question 5

Evaluate the following real integrals:

- $\displaystyle\int_0^\infty \frac{dx}{x^4 + x^2 + 1}$
- $\displaystyle\int_0^\infty \frac{\cos(4x)dx}{(x^2 + 1)^2}.$

Question 6

Find a bilinear transformation, $\mathcal{T}$ which maps the circle $|z - 1| = 2$ onto the line $x + y = 1$.

Let $P(t) = 1 + 2e^{it}$ parametrize the circle $|z - 1| = 2$, for t real.
Sketch the four points $P_k = P\left(\frac{k\pi}{3}\right)$, $k = 1, 2, 4, 5$.
Find and sketch the images Q_k of these points.
Compute the cross-ratio of the points on the circle and of their images and verify that these cross-ratios are equal.