Complex variables: Exam 1 Solutions 2/25/10

Question 1

Compute the following limits, or explain why the limit in question does not exist.

- \[ A = \lim_{z \to 1+i} \left( \frac{z^4 + 4}{z^2 - 3iz - 3 + i} \right) \]

Note that \((1 + i)^2 = 1 - 1 + 2i = 2i\).

Then we have:

\[ z^4 + 4 = (z^2)^2 - (2i)^2 = (z^2 - 2i)(z^2 + 2i) = (z - (1 + i))(z + 1 + i)(z^2 + 2i). \]

Also we have (since \((1 + i)(2i - 1) = 2i - 2 - 1 - i = -3 + i\)):

\[ z^2 - 3iz - 3 + i = (z - (1 + i))(z - 2i + 1) \]

So we get:

\[ A = \lim_{z \to 1+i} \left( \frac{z^4 + 4}{z^2 - 3iz - 3 + i} \right) \]

\[ = \lim_{z \to 1+i} \left( \frac{(z - (1 + i))(z + 1 + i)(z^2 + 2i)}{(z - (1 + i))(z - 2i + 1)} \right) \]

\[ = \lim_{z \to 1+i} \left( \frac{(z + 1 + i)(z^2 + 2i)}{(z - 2i + 1)} \right) \]

\[ = \frac{(1 + i + 1 + i)(2i + 2i)}{(1 + i - 2i + 1)} \]

\[ = \frac{(2 + 2i)(4i)}{2 - i} = \frac{(-8 + 8i)(2 + i)}{5} \]

\[ = \frac{-16 - 8 + 16i - 8i}{5} = \frac{-24 + 8i}{5}. \]
Alternatively, we first note that this is a "0/0" limit, since we have, when \( z = 1 + i \):

\[
\begin{align*}
z &= 1 + i, \\
z^2 &= 2i, \\
z^3 &= z^2 z = 2i(1 + i) = -2 + 2i, \\
z^4 &= (z^2)^2 = (2i)^2 = -4, \\
z^4 + 4 &= 0, \\
z^2 - 3iz - 3 + i &= 2i - 3i(1 + i) - 3 + i = 2i - 3i + 3 - 3 + i = 0.
\end{align*}
\]

Then we can use L'Hôpital:

\[
A = \lim_{z \to 1+i} \left( \frac{z^4 + 4}{z^2 - 3iz - 3 + i} \right) \\
= \lim_{z \to 1+i} \left( \frac{4z^3}{2z - 3i} \right) \\
= \frac{4(-2 + 2i)}{2(1 + i) - 3i} \\
= \frac{-8 + 8i}{2 - i} = \frac{(-8 + 8i)(2 + i)}{5} \\
= \frac{-16 - 8 + 16i - 8i}{5} = \frac{-24 + 8i}{5}.
\]
\[ B = \lim_{z \to 0} \left( \frac{\Re(z) \Im(z)}{z} \right). \]

Put \( z = re^{it} \), with \( r > 0 \) and \( t \) real. Then we have:

\[ z = re^{it} = r \cos(t) + ir \sin(t) \]
\[ \Re(z) = r \cos(t), \Im(z) = r \sin(t), \]
\[ B = \lim_{z \to 0} \left( \frac{\Re(z) \Im(z)}{z} \right) \]
\[ = \lim_{r \to 0^+} \left( \frac{r \cos(t) r \sin(t)}{re^{it}} \right) \]
\[ = \lim_{r \to 0^+} r \sin(t) \cos(t) e^{-it}. \]

But the factor \( \sin(t) \cos(t) e^{-it} \) has size:

\[ |\sin(t) \cos(t) e^{-it}| = |\sin(t) \cos(t)| = \frac{1}{2} |\sin(2t)| \leq \frac{1}{2}. \]

So since in the expression of the limit, the factor \( r \) goes to zero and the factor \( \sin(t) \cos(t) e^{-it} \) is bounded in size, the limit exists and is zero. So \( B = 0 \).

\[ C = \lim_{z \to 3i} \frac{|z - 3i|^2}{z^2 + 9} \]

Put \( z = 3i + re^{it} \), where \( r > 0 \) and \( t \) is real. Then we have:

\[ C = \lim_{z \to 3i} \frac{|z - 3i|^2}{z^2 + 9} = \lim_{z \to 3i} \frac{|z - 3i|^2}{(z - 3i)(z + 3i)} \]
\[ = \frac{1}{6i} \lim_{z \to 3i} \frac{|z - 3i|^2}{z - 3i} = \frac{1}{6i} \lim_{r \to 0^+} \frac{|re^{it}|^2}{re^{it}} \]
\[ = \frac{1}{6i} \lim_{r \to 0^+} re^{-it}. \]

So since in the expression of the limit, the factor \( r \) goes to zero and the factor \( e^{-it} \) is bounded in size, having size \( |e^{-it}| = 1 \), the limit exists and is zero. So \( C = 0 \).
Question 2

Determine the images of the circle of radius 5 and center $-5 + 3i$ and of the line $y = -x$ under the transformation $z \rightarrow \frac{1}{z + 1}$.

Plot the curves and their images under the transformation on one graph.

The inverse transformation is given by solving the following equation for $w$ in terms of $z$:

$$z = \frac{1}{w + 1}, \quad w + 1 = \frac{1}{z},$$

$$w = \frac{1}{z} - 1 = \frac{1 - z}{z}.$$

The circle of radius 5 and center $-5 + 3i$ has the equation:

$$|z + 5 - 3i| = 5.$$

Its image under the given transformation is obtained by substituting the formula for the inverse transformation into this equation, so is:

$$\left|\frac{1 - z}{z} + 5 - 3i\right| = 5,$$

$$|1 - z + 5z - 3iz| = 5|z|,$$

$$|1 + 4z - 3iz| = 5|z|,$$

$$|1 + (4 - 3i)z|^2 = 25|z|^2,$$

$$(1 + (4 - 3i)z)(1 + (4 + 3i)\overline{z}) = 25z\overline{z},$$

$$1 + (4 - 3i)z + (4 + 3i)\overline{z} + 25z\overline{z} = 25z\overline{z},$$

$$1 + (4 - 3i)z + (4 + 3i)\overline{z} = 0.$$

Putting $z = x + iy$, with $x$ and $y$ real, we get:

$$0 = 1 + (4 - 3i)(x + iy) + (4 + 3i)(x - iy) = 1 + 4x + 3y - 3ix + 4iy + 4x + 3y + 3ix - 4iy,$$

$$8x + 6y + 1 = 0.$$

This is a straight line through the point $\left(0, -\frac{1}{6}\right)$ with slope $-\frac{4}{3}$.

Note that the image of the given circle is a straight line rather than a circle, because the point $-1$ is sent to infinity by the transformation, and the point $-1$ lies on the given circle, since $| -1 + 5 - 3i| = |4 - 3i| = 5$.  

4
The line $y = -x$ has the equation: $\mathcal{R}(z) + \mathfrak{I}(z) = 0$.
Its image is then:

$$\mathcal{R} \left( \frac{1 - z}{z} \right) + \mathfrak{I} \left( \frac{1 - z}{z} \right) = 0.$$ 

We simplify by multiplying by the real quantity $z\overline{z}$, which can be brought inside the $\mathcal{R}$ and $\mathfrak{I}$ terms, since $\mathcal{R}(ua) = u\mathcal{R}(a)$ and $\mathfrak{I}(ua) = u\mathfrak{I}(a)$, for any real $u$ and any complex number $a$.

$$\mathcal{R} \left( (1 - z)\overline{z} \right) + \mathfrak{I} \left( (1 - z)\overline{z} \right) = 0,$$

$$\mathcal{R} \left( \overline{z} - z\overline{z} \right) + \mathfrak{I} \left( (\overline{z} - z\overline{z}) \right) = 0,$$

$$\mathcal{R}(z) - z\overline{z} + \mathfrak{I}(z) = 0,$$

$$0 = z\overline{z} - \mathcal{R}(z) + \mathfrak{I}(z).$$

Here we used that $\mathcal{R}(a + b) = \mathcal{R}(a) + \mathcal{R}(b)$, $\mathfrak{I}(a + b) = \mathfrak{I}(a) + \mathfrak{I}(b)$, for any complex numbers $a$ and $b$.

Also we used that, for any real $u$, $\mathcal{R}(u) = u$ and $\mathfrak{I}(u) = 0$.

Also we used that $\mathcal{R}(a) = \mathcal{R}(\overline{a})$ and $\mathfrak{I}(a) = -\mathfrak{I}(a)$, for any complex $a$.

Continuing we get:

$$0 = z\overline{z} - \frac{1}{2}(z + \overline{z}) + \frac{1}{2i}(z + \overline{z}) = z\overline{z} - \left( \frac{1 + i}{2} \right) z - \left( \frac{1 - i}{2} \right) \overline{z}$$

$$= \left( z - \frac{1 - i}{2} \right) \left( \overline{z} - \frac{1 + i}{2} \right) - \frac{1 - i)(1 + i)}{4}$$

$$= \left| z - \frac{1}{2}(1 - i) \right|^2 - \frac{1}{2},$$

$$\left| z - \frac{1}{2}(1 - i) \right| = \frac{1}{\sqrt{2}}.$$ 

So the image of the given line is a circle center the point $\frac{1 - i}{2}$ and radius $\frac{1}{\sqrt{2}}$.

Alternatively, we write $z = x + iy$, with $x$ and $y$ real.

Then the required equation is:

$$0 = z\overline{z} - \mathcal{R}(z) + \mathfrak{I}(z) = x^2 + y^2 - x + y = \left( x - \frac{1}{2} \right)^2 + \left( y + \frac{1}{2} \right)^2 - \frac{1}{2}.$$ 

Again this is the equation of a circle, centered at $\left( \frac{1}{2}, -\frac{1}{2} \right)$, with radius $\frac{1}{\sqrt{2}}$, in agreement with the above.
Question 3
Let \( u = x^2 - y^2 - 2xy + 6x^2y + 3xy^2 - 2y^3 - x^3 \).

- Prove that \( u \) is harmonic and find its harmonic conjugate \( v \).

We solve the Cauchy-Riemann equations.
- The equation \( v_x = -u_y \) gives:
  \[
v_x = -u_y = 2y + 2x - 6x^2 - 6xy + 6y^2,
  \]
  \[
v = \int (2y + 2x - 6x^2 - 6xy + 6y^2)dx = 2xy + x^2 - 2x^3 - 3x^2y + 6xy^2 + g(y).
  \]
- Then the equation \( 0 = v_y - u_x \) gives:
  \[
  0 = g'(y) + 2x - 3x^2 + 12xy - 2x + 2y - 12xy - 3y^2 + 3x^2
  = g'(y) + 2y - 3y^2.
  \]
  \[
g = \int (3y^2 - y^2)dy = y^3 - y^2 + C.
  \]

So the harmonic conjugate \( v \) of \( u \) exists and is:
\[
v = 2xy + x^2 - 2x^3 - 3x^2y + 6xy^2 - y^2 + y^3 + C.
\]

In particular, it follows that \( u \) is harmonic:
\[
u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0.
\]

Alternatively, we use differentials.
We need:
\[
dv = v_x dx + v_y dy = -u_y dx + u_x dy
= (2y + 2x - 6x^2 - 6xy + 6y^2)dx + (2x - 2y + 12xy + 3y^2 - 3x^2)dy
= 2xdx - 6x^2 dx - 2ydy + 3y^2 dy + (2ydx + 2xdy) - (6ydx + 3x^2 dy) + (6y^2 dx + 12xy dy)
= d(x^2 - 2x^3 - y^2 + y^3 + 2xy - 3x^2y + 6xy^2).
\]
So \( v = x^2 - 2x^3 - y^2 + y^3 + 2xy - 3x^2y + 6xy^2 + C \), as before.
• Express \( f = u + iv \) as a function of \( z = x + iy \) and show that:

\[
f'(z) = \frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}.
\]

We have:

\[
u(x, y) + iv(x, y) = f(x + iy),
\]

\[
f(x) = u(x, 0) + iv(x, 0) = x^2 - x^3 - iy(x^2 - 2x^3 + C) = x^2(1 + i) - x^3(1 + 2i).
\]

So \( f(z) = z^2(1 + i) - z^3(1 + 2i) + iC \).

Check:

\[
z^2(1 + i) - z^3(1 + 2i) + iC
\]

\[
(x^2 - y^2 + 2ixy)(1 + i) - (x^3 - 3xy^2 + 3ix^2y - iy^3)(1 + 2i) + iC
\]

\[
= x^2 - y^2 - 2xy - x^3 + 3xy^2 + 6x^2y - 2y^3 + i(2xy + x^2 - y^2 - 2x^3 + 6xy^2 - 3x^2y - 6y^3 + C)
\]

\[
= u + iv.
\]

Then we have:

\[
f'(z) = 2z(1 + i) - 3z^2(1 + 2i)
\]

\[
= 2(x + iy)(1 + i) - 3(x^2 - y^2 + 2ixy)(1 + 2i)
\]

\[
= 2x - 2y - 3x^2 + 3y^2 + 12xy + i(2x + 2y - 6x^2 + 6y^2 - 6xy).
\]

Also we have:

\[
f_x = u_x + iv_x = 2x - 2y + 12xy + 3y^2 - 3x^2 + i(2y + 2x - 6x^2 - 6xy + 6y^2) = f'(z),
\]

\[
-if_y = v_y - iu_y = 2x - 3x^2 + 12xy - 2y + 3y^2 + i(2y + 2x - 6x^2 - 6xy + 6y^2) = f_x = f'(z).
\]

So we have \( f'(z) = f_x = -if_y \), as required.
Question 4

Let \( f(z) = \frac{z^2 + 1}{z} \).

- Find by computing an appropriate limit, the complex derivative of \( f(z) \). What is the domain of \( f'(z) \)? Explain your answer.

- When \( z = x + iy \), with \( x \) and \( y \) real, write \( f = u + iv \), where \( u(x, y) \) and \( v(x, y) \) are real functions. Show that \( u \) and \( v \) are harmonic and that:

\[
f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.
\]

We have, when \( z \neq 0 \):

\[
f'(z) = \lim_{w \to z} \left( \frac{f(w) - f(z)}{w - z} \right)
\]

\[
= \lim_{w \to z} \left( \frac{w^2 + 1 - \frac{z^2 + 1}{z}}{w - z} \right)
\]

\[
= \lim_{w \to z} \frac{z(w^2 + 1) - w(z^2 + 1)}{wz(w - z)}
\]

\[
= \frac{1}{z^2} \lim_{w \to z} \frac{zw^2 - wz^2 + z - w}{w - z}
\]

\[
= \frac{1}{z^2} \lim_{w \to z} \frac{zw(w - z) - 1(w - z)}{w - z}
\]

\[
= \frac{1}{z^2} \lim_{w \to z} (zw - 1)
\]

\[
= \frac{z^2 - 1}{z^2}.
\]

The domain of \( f' \) is the same as that of \( f \), namely all non-zero complex numbers.
Next we have:

$$f(z) = \frac{z^2 + 1}{z} = z + \frac{1}{z} = x + iy + \frac{x - iy}{x^2 + y^2} = u + iv,$$

$$u = x + \frac{x}{x^2 + y^2}, \quad v = y - \frac{y}{x^2 + y^2}.$$

Then we have:

$$u_x = 1 + \frac{1}{x^2 + y^2} - \frac{2x^2}{(x^2 + y^2)^2} = 1 + \frac{y^2 - x^2}{(x^2 + y^2)^2},$$

$$u_y = -\frac{2xy}{(x^2 + y^2)^2},$$

$$v_x = \frac{2xy}{(x^2 + y^2)^2},$$

$$v_y = 1 - \frac{1}{x^2 + y^2} + \frac{2y^2}{(x^2 + y^2)^2} = 1 + \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

Then $u_x = v_y$ and $u_y = -v_x$, so $u$ and $v$ are harmonic:

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0,$$

$$v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0.$$

Finally we have, since $\overline{z^2} = x^2 + y^2$ and $\overline{(x - iy)^2} = (x^2 - y^2 - 2ixy)$:

$$u_x + iv_x = 1 + \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{2ixy}{(x^2 + y^2)^2}$$

$$= 1 - \frac{x^2 - y^2 - 2ixy}{(z\overline{z})^2}$$

$$= 1 - \frac{x^2}{z^2\overline{z^2}}$$

$$= 1 - \frac{1}{z^2}$$

$$= \frac{z^2 - 1}{z^2} = f'(z).$$

So $f'(z) = u_x + iv_x$, as required.
Question 5

Consider the transformation \( T : z \to (2 - 2i)z - 12 - i \), defined for any \( z \in \mathbb{C} \).

- Describe the transformation geometrically.

This is an affine linear transformation of the complex plane to itself. We have \( 2 - 2i = 2\sqrt{2}e^{-\frac{i\pi}{4}} \), so the transformation is a rotation through forty-five degrees, clockwise about the origin, followed by a dilation centered at the origin by a factor of \( 2\sqrt{2} \), followed by a translation through the vector \([-12, -1]\), so twelve units to the left and one unit down.

- Find the fixed points of the transformation, if any.

We solve the equation:

\[
0 = T(z) - z = (2 - 2i)z - 12 - i - z = (1 - 2i)z - 12 - i,
\]

\[
(1 - 2i)z = 12 + i,
\]

\[
(1 + 2i)(1 - 2i)z = (1 + 2i)(12 + i),
\]

\[
5z = 10 + 25i,
\]

\[
z = 2 + 5i.
\]

Check:

\[
T(2 + 5i) = (2 - 2i)(2 + 5i) - 12 - i = 4 + 10 - 4i + 10i - 12 - i = 2 + 5i.
\]

So \( z = 2 + 5i \) is the unique fixed point of the transformation \( T \).

- Find a formula for the inverse transformation.

We solve the equation \( z = T(w) \), for \( w \) in terms of \( z \):

\[
z = (2 - 2i)w - 12 - i,
\]

\[
(2 - 2i)w = z + 12 + i,
\]

\[
(2 + 2i)(2 - 2i)w = (2 + 2i)(z + 12 + i),
\]

\[
8w = (2 + 2i)z + 22 + 26i,
\]

\[
w = \frac{1}{4}((1 + i)z + 11 + 13i).
\]
So the inverse transformation is defined for any complex $z$ and is:

$$T^{-1}(z) = \frac{1}{4}((1 + i)z + 11 + 13i).$$

Check:

- First we have:

$$T(T^{-1}(z)) = (2 - 2i)T^{-1}(z) - 12 - i$$

$$= \frac{1}{4}(2 - 2i)((1 + i)z + 11 + 13i) - 12 - i$$

$$= \frac{1}{2}((1 - i)(1 + i)z + (1 - i)(11 + 13i)) - 12 - i$$

$$= \frac{1}{2}(2z + 24 + 2i) - 12 - i = z.$$

- Next we have:

$$T^{-1}(T(z)) = \frac{1}{4}((1 + i)T(z) + 11 + 13i)$$

$$= \frac{1}{4}((1 + i)((2 - 2i)z - 12 - i) + 11 + 13i)$$

$$= \frac{1}{4}(4z - 12 + 1 - i - 12i + 11 + 13i) = z.$$

Since, for any complex $z$, we have $T(T^{-1}(z)) = T^{-1}(T(z)) = z$, the map $T$ is invertible, with inverse the map $T^{-1}$. 

11
• Find the images under the transformation of the lines $\mathcal{L}$ and $\mathcal{M}$ with the following parametric equations, where $s$ and $t$ are real parameters:

$$\mathcal{L} : z = i + (1 + 2i)s \quad \text{and} \quad \mathcal{M} : z = (2 - i)t.$$  

Also sketch the lines $\mathcal{L}$ and $\mathcal{M}$ and their images on the complex plane.

- The image of $\mathcal{L}$ under $T$ is the curve, for $s$ real:

$$T(i + (1 + 2i)s) = (2 - 2i)(i + (1 + 2i)s) - 12 - i$$

$$= 2i + 2 + (2 + 4i - 2i + 4)s - 12 - i$$

$$= (6 + 2i)s + i - 10.$$  

This is a straight-line through the point $(-10, 1)$ with the direction vector $(3, 1)$, so slope $\frac{1}{3}$.

It has the Cartesian equation:

$$0 = x - 3y + 13.$$  

- The image of $\mathcal{M}$ under $T$ is the curve, for $t$ real:

$$T((2 - i)t) = (2 - 2i)(2 - i)t - 12 - i$$

$$= (2 - 6i)t - 12 - i.$$  

This is a straight-line through the point $(-12, -1)$ with the direction vector $(1, -3)$, so slope $-3$.

It has the Cartesian equation:

$$0 = 3x + y + 37.$$  

Note that the lines $\mathcal{L}$ and $\mathcal{M}$ are perpendicular, since their slopes are $2$ and $-\frac{1}{2}$, respectively, as are their images, with slopes $\frac{1}{3}$ and $-3$, respectively.
• Find the image under the transformation of the circle $|z - 2| = 5$ and sketch the circle and its image on the complex plane.

The given circle has parametric equation $z = 2 + 5e^{it}$, with $t$ real. Its image under $T$ is then the curve:

$T(2 + 5e^{it}) = (2 - 2i)(2 + 5e^{it}) - 12 - i$

$= 4 - 4i - 12 - i + 10(1 - i)e^{it}$

$= -8 - 5i + 10\sqrt{2}e^{i(t-\pi/4)}$

$= -8 - 5i + 10\sqrt{2}e^{is}, \quad s = t - \frac{\pi}{4}$.

As $t$ ranges over the real line, so does $s$, so the image is the circle of radius $10\sqrt{2}$ centered at $-8 - 5i$.

So its equation is $|z + 8 + 5i| = 10\sqrt{2}$.

Its Cartesian equation is then:

$$(x + 8)^2 + (y + 5)^2 = 200,$$

$$x^2 + y^2 + 16x + 10y = 111.$$ 

Note that both circles go through the fixed point $2 + 5i$ of $T$, since $|2 + 5i - 2| = |5i| = 5$ and $|2 + 5i + 8 + 5i| = |10 + 10i| = 10\sqrt{2}.$
Question 6

Consider the function \( f(z) = (z - 1)^{\frac{1}{2}}(z + 1)^{\frac{1}{2}} \).

- If \( f(0) = i \) and the branch cuts for \( f \) are the intervals \([1, \infty)\) and \((-\infty, -1]\) along the real axis, determine the values of \( f \):
  - when \( z = 9 \) is approached from the upper half-plane.
  - when \( z = 9 \) is approached from the lower half-plane.

- If instead \( f(0) = i \) and the branch cuts for \( f \) are the vertical lines in the upper-half plane starting at the points \( z = \pm 1 \), explain why \( f(9) \) is unambiguous and determine its value.

Let \( z - 1 = pe^{is} \) and \( z + 1 = qe^{it} \), where \( p, q, s \) and \( t \) are real and \( p \) and \( q \) are positive (so we avoid the points \( z = \pm 1 \). Then we have:

\[
f(z) = (pq)^{\frac{1}{2}}e^{\frac{i}{2}(s+t)}.
\]

At the point \( z = 0 \), we have \( z - 1 = -1 \) and \( z + 1 = 1 \), so \( p = q = 1 \) and we may take: \( s = \pi \) and \( t = 0 \). This gives the required value:

\[
f(0) = ((1)(1))^{\frac{1}{2}}e^{\frac{i}{2}((\pi + 0))} = i.
\]

The freedom in choice of the pair \((s, t)\) is then:

\[
(s, t) = ((2k + 1)\pi, -2k\pi + 4m\pi).
\]

Here \( k \) and \( m \) are any integer.

First suppose that the cuts for \( f \) are the intervals \([1, \infty)\) and \((-\infty, -1]\) along the real axis.

At the point \( z = 9 \), we have \( z - 1 = 8 \) and \( z + 1 = 10 \), so \( p = 8 \) and \( q = 10 \). Also moving continuously from the origin to 9 in the upper half-plane, the angle \( s \) decreases to \( 2k\pi \), whereas the angle \( t \) returns to its initial value. So we get in the limit:

\[
f(9) = ((8)(10))^{\frac{1}{2}}e^{\frac{i}{2}(2k\pi - 2k\pi + 4m\pi)} = 4\sqrt{5}.
\]

14
If instead, we move continuously from the origin to 9 in the lower half-plane, the angle $s$ increases to $(2k + 2)\pi$, whereas the angle $t$ again returns to its initial value. So we get in the limit:

$$f(9) = ((8)(10))^{\frac{1}{2}}e^{\frac{i}{2}(2k+2)\pi-2k\pi+4m\pi} = -4\sqrt{5}.$$ 

Next suppose that the cuts for $f$ are the vertical lines in the upper-half plane starting at the points $z = \pm 1$.

At the point $z = 9$, we have $z - 1 = 8$ and $z + 1 = 10$, so $p = 8$ and $q = 10$. Also moving continuously from the origin to 9 avoiding the cuts, so going under the cut at $z = 1$, the angle $s$ increases to $(2k + 2)\pi$, whereas the angle $t$ returns to its initial value. So we get the unambiguous value for $f(9)$.

$$f(9) = ((8)(10))^{\frac{1}{2}}e^{\frac{i}{2}(2k+2)\pi-2k\pi+4m\pi} = -4\sqrt{5}.$$ 

In general, the cuts entail that the angles $s$ and $t$ are unambiguous, anywhere except on the cuts, given their initial values at some point off the cut. Then the function is itself unambiguous, since at the initial point $(s, t)$ are given up to $(s + 2k\pi, t - 2k\pi + 4m\pi)$, where $k$ and $m$ are integers. Then, by continuity, there is the same ambiguity in the angles at every point and this ambiguity does not affect the value of $f(z)$, since the angle $\frac{1}{2}(s + t)$ changes by an integer multiple of $2\pi$. 

15
Question 7

Consider the function \( f(z) = \ln(z - i) - \ln(z + i) \), where the branch cut goes from \( z = i \) to \( z = -i \) along the semi-circle \( |z| = 1 \), with \( \Re(z) \leq 0 \).

- If \( f(0) = -\pi i \), determine the possible values of \( f(3) \) and \( f(-3) \).

Let \( z - i = pe^{is} \) and \( z + i = qe^{it} \), where \( p, q, s \) and \( t \) are real and \( p \) and \( q \) are positive (so we avoid the points \( z = \pm i \)).

Then \( f(z) = \ln(p) - \ln(q) + i(s - t) \).

At \( z = 0 \), we may take \( z - i = -i = e^{-i\pi} \) and \( z + i = i = e^{i\pi} \).

This gives \( p = q = 1 \) and allows us to take \( s = -\frac{\pi}{2} \) and \( t = \frac{\pi}{2} \), giving:

\[
f(0) = \ln(1) - \ln(1) + i \left( -\frac{\pi}{2} - \frac{\pi}{2} \right) = -i\pi.
\]

This gives us the required value for \( f(0) \).

The remaining freedom in the choice of the angles \( s \) and \( t \), keeping the correct value of \( f(0) \) is \( (s, t) = \left( -\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi \right) \), where \( k \) is any integer.

Now moving to the point \( z = 3 \), the vector \( z - i \) is the vector \( [3, -1] \), whereas the vector \( z + i \) is the vector \( [3, 1] \), so we get \( p = q = \sqrt{10} \) and \( s \) increases to \(-\arctan \left( \frac{1}{3} \right) + 2k\pi \), whereas \( t \) decreases to \( \arctan \left( \frac{1}{3} \right) + 2k\pi \).

So we get a unique value for \( f(3) \):

\[
f(3) = \ln \left( \sqrt{10} \right) - \ln \left( \sqrt{10} \right) + i \left( - \arctan \left( \frac{1}{3} \right) + 2k\pi - \arctan \left( \frac{1}{3} \right) - 2k\pi \right)
\]

\[
= -2i \arctan \left( \frac{1}{3} \right).
\]
Moving continuously to the point \( z = -3 \), the vector \( z - i \) is the vector \([-3, -1]\), whereas the vector \( z + i \) is the vector \([-3, 1]\), so, if we go around over the top of the cut, we get \( p = q = \sqrt{10} \) and \( s \) increases to \( \arctan \left( \frac{1}{3} \right) + 2k\pi \), whereas \( t \) decreases to \(- \arctan \left( \frac{1}{3} \right) + 2k\pi \).

Alternatively, if we go around the bottom of the cut, \( s \) decreases to \( \arctan \left( \frac{1}{3} \right) + (2k - 2)\pi \) and \( t \) decreases to \( \arctan \left( \frac{1}{3} \right) + (2k - 2)\pi \). In either case, we get the same value for \( f(-3) \):

\[
f(-3) = \ln \left( \sqrt{10} \right) - \ln \left( \sqrt{10} \right) + i \left( \arctan \left( \frac{1}{3} \right) + 2k\pi + \arctan \left( \frac{1}{3} \right) - 2k\pi \right)
\]

\[
= \ln \left( \sqrt{10} \right) - \ln \left( \sqrt{10} \right) + i \left( \arctan \left( \frac{1}{3} \right) + (2k - 2)\pi + \arctan \left( \frac{1}{3} \right) - (2k - 2)\pi \right)
\]

\[
= 2i \arctan \left( \frac{1}{3} \right).
\]

Indeed, away from the cut, the function \( f(z) \) is everywhere well-defined and analytic.

- Determine the complex derivative of \( f \) (you may assume that this derivative exists).

The derivative of \( f(z) \) is just given by the standard rules of calculus:

\[
f'(z) = \frac{1}{z - i} \cdot 1 - \frac{1}{z + i} \cdot 1 = \frac{z + i - (z - i)}{(z - i)(z + i)} = \frac{2i}{z^2 + 1}.
\]

Note that the function \( \arctan(z) \) has derivative \( \frac{1}{z^2 + 1} \), so the functions \( \frac{f(z)}{2i} \) and \( \arctan(z) \) differ at most by a constant.