Homework 7, Question 1

Use Euler’s method with step size $h = 0.1$ to compute the first five iterations for the differential equation:

$$y' = t + y, \quad y(0) = 1.$$ 

Plot your results.

Also determine and plot the exact solution and compare your results with the exact solution.

The update formula is:

$$(t, y) \rightarrow (t + h, y(t) + hy'(t)) = (t + h, y + h(t + y)) = (t + h, y(1 + h) + ht).$$

So when $h = 0.1$, we get:

$$(t, y) \rightarrow (t + 0.1, \frac{1}{10}(11y + t)).$$

$$(0, 1) \rightarrow (0.1, 1.1),$$

$$(0.1, 1.1) \rightarrow (0.2, \frac{1}{10}(12.1 + 0.1) = (0.2, 1.22),$$

$$(0.2, 1.22) \rightarrow (0.3, \frac{1}{10}(13.42 + 0.2) = (0.3, 1.362),$$

$$(0.3, 1.362) \rightarrow (0.4, \frac{1}{10}(14.982 + 0.3) = (0.4, 1.5282),$$

$$(0.4, 1.5282) \rightarrow (0.5, \frac{1}{10}(16.8102 + 0.4) = (0.5, 1.72102).$$

The particular solution is $y = -1 - t$, so the general solution is $y = -t - 1 + Ae^t$.

Then $1 = -1 + A$, so $A = 2$ and the exact solution is $y = 2e^t - t - 1$.

This gives the points:

$$(0, 1), (0.1, 1.103), (0.2, 1.2428), (0.3, 1.3997), (0.4, 1.5836), (0.5, 1.7974).$$

We see that the exact solution is systematically above the numerical solution.

One way to see this is that $y'' = (t + y)' = 1 + y' = 1 + t + y$ is always positive for this solution, so the trajectory is always curving up and away from its linearized approximation.
Homework 7, Question 4

Find the general solution of the following system and discuss the resulting trajectories:

\[ x' = -x^3y^2, \quad y' = -\frac{1}{2}(x - 2)y, \quad x(0) = \frac{1}{2}, \quad y(0) = 2. \]

(Hint: first obtain an equation for \( y \) in terms of \( x \)).

We have:

\[ \frac{dy}{dx} = \frac{y'}{x'} = -\frac{1}{2}(x - 2)\frac{y}{-x^3y^2}, \]

\[ 2ydy = \frac{(x - 2)dx}{x^3} = \frac{(x^{-2} - 2x^{-3})dx}{x}, \]

\[ y^2 = C - x^{-1} + x^{-2}. \]

When \( x = \frac{1}{2} \), we have \( y = 2 \), so we get:

\[ 4 = C - 2 + 4, \quad C = 2, \]

\[ y^2 = 2 - x^{-1} + x^{-2} = \frac{2x^2 - x + 1}{x^2}, \]

\[ y = x^{-1}\sqrt{2x^2 - x + 1}. \]

Also we have:

\[ x^2(y^2 - 2) + x - 1 = 0, \]

\[ x = \frac{-1 + \sqrt{4y^2 - 7}}{2y^2 - 4}. \]

\[ -2y^{-1}\frac{dy}{dt} = x - 2 = \frac{7 - 4y^2 + \sqrt{4y^2 - 7}}{2y^2 - 4}, \]

\[ t + A = -\int \frac{(2y^2 - 4)dy}{y^2\left(7 - 4y^2 + \sqrt{4y^2 - 7}\right)}. \]

Put \( 4y^2 - 7 = u^2 \), so \( u = \pm \sqrt{4y^2 - 7} = \frac{2 - x}{x} \).

In the region \( 0 < x < 2 \), we have \( u \) positive, \( u = \sqrt{4y^2 - 7} \).

In the region \( x \geq 2 \), we have \( u = -\sqrt{4y^2 - 7} \).

At \( x = 2 \), \( u = 0 \) and \( y = \frac{\sqrt{7}}{2} \).
When \( t = 0 \), we have \( u = 3 \).
Then \( 4y \, dy = u \, du \) and the integral becomes:

\[
t + A = \int \frac{(1 - u^2) \, du}{(u^2 + 7)(u - u^2)} = \int \frac{(1 + u) \, du}{(u^2 + 7)} = \frac{1}{\sqrt{7}} \arctan \left( \frac{u}{\sqrt{7}} \right) + \frac{1}{2} \ln(u^2 + 7).
\]

So we get:

\[
t = \frac{1}{\sqrt{7}} \arctan \left( \frac{u}{\sqrt{7}} \right) - \frac{1}{\sqrt{7}} \arctan \left( \frac{3}{\sqrt{7}} \right) + \frac{1}{2} \ln(u^2 + 7) - 2 \ln(2)
\]

\[
= \pm \frac{1}{\sqrt{7}} \arctan \left( \sqrt{\frac{4y^2 - 7}{7}} \right) - \frac{1}{\sqrt{7}} \arctan \left( \frac{3}{\sqrt{7}} \right) + \ln(y) - \ln(2)
\]

\[
= \frac{1}{\sqrt{7}} \arctan \left( \frac{2 - x}{x \sqrt{7}} \right) - \frac{1}{\sqrt{7}} \arctan \left( \frac{3}{\sqrt{7}} \right) + \frac{1}{2} \ln(2x^2 - x + 1) - 2 \ln(2x)).
\]

For the plot forward in time, we see that \( x \) decreases towards zero, approaching \( x = 0 \) asymptotically, as \( y \) goes to infinity.
Going backwards in time, \( y \) decreases to a minimum when \( x = 2 \) at the point

\[
(x, y, t) = \left( 2, \frac{\sqrt{7}}{2}, -\frac{1}{2} \ln \left( \frac{16}{7} \right) - \frac{1}{\sqrt{7}} \arctan \left( \frac{3}{\sqrt{7}} \right) \right) = (2, 1.3229, -0.6324).
\]

From then on \((x, y)\) both increase asymptotically going to \( y = \left( \sqrt{2} \right)^{-1} \), as \( x \to \infty \) and \( t \) goes to the value:

\[
-\frac{1}{\sqrt{7}} \arctan \left( \sqrt{7} \right) - \frac{1}{2} \ln(2) = -0.8037.
\]

So the motion ends in finite time in the past.

So the motion starts out at \((x, y, t) = (\infty, \left( \sqrt{2} \right)^{-1}, -0.8037)\), \( x \) and \( y \) decrease until \( y \) reaches its minimum value as discussed above and then \( y \) increases to infinity as \( x \to 0^+ \) and \( t \to \infty \).
The curve of \( y \) against \( x \) is concave up until \((x, y, t) = (2.9847, 1.3331, -0.7731)\) and is concave down for \( x > 2.9847 \) and \(-0.8037 < t < -0.7731\).
Homework 7, Question 6

Solve the system: \( X' = AX + F \), with initial condition \( X(0) \), given as follows:

\[
A = \begin{pmatrix}
-1 & 1 \\
-1 & -1
\end{pmatrix}, \quad F = \begin{pmatrix}
5 \cos(t) \\
10 \sin(t)
\end{pmatrix}, \quad X(0) = \begin{pmatrix}
3 \\
-1
\end{pmatrix}
\]

Plot the solution and discuss its behavior.

We have:

\[ \det(A) = 2, \quad \text{tr}(A) = -2, \quad A^2 + 2A + 2I = 0, \quad (A + I)^2 = -I, \]

Put \( J = A + I \). Then we have:

\[
J = A + I = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix},
\]

\[ J^2 = -I, \quad M(t) = e^{Jt} = \cos(t)I + J\sin(t). \]

Check:

\[ M(0) = I, \]

\[ M'(t) - JM = -\sin(t)I + J\cos(t) - J(\cos(t)I + J\sin(t)) \]

\[ = -\sin(t)I - J^2\sin(2t) = 0. \]

Then we have:

\[
G(t) = e^{At} = e^{-t}e^{(A+I)t} = e^{-t}e^{Jt} = e^{-t}(\cos(t)I + J\sin(t)) = e^{-t} \begin{pmatrix}
\cos(t) & \sin(t) \\
-\sin(t) & \cos(t)
\end{pmatrix}.
\]

Note that \( G(t) \) can be rewritten:

\[ 2e^tG(t) = 2\cos(t)I + 2J\sin(t) = e^{it}I + e^{-it})I - i(e^{it} - e^{-it})J \]

\[ = e^{it}(I - iJ) + e^{-it}(I + iJ), \]

\[ G(t) = \frac{e^{-t}}{2} (e^{it}(I - iJ) + e^{-it}(I + iJ)). \]

Then a particular solution is \( X = G(t)K(t) \), where:

\[
K'(t) = G(-t)F(t) = \frac{5e^t}{2} \Re \left( (e^{-it}(I - iJ) + e^{it}(I + iJ)) \begin{pmatrix}
e^{it} \\
-2ie^{it}
\end{pmatrix} \right)
\]

\[ = \frac{5}{2} \Re \left( (I - iJ)e^t + e^{(1+2i)t}(I + iJ) \begin{pmatrix}1 \\
-2i
\end{pmatrix} \right). \]
Integrating, ignoring integration constants, we get:

\[
K(t) = \frac{1}{2} \Re \left( 5(I - iJ)e^t + (1 - 2i)e^{(1+2i)t}(I + iJ) \right) \bigg|_{-2i}^{1}.
\]

Then we have:

\[
X_p = G(t)K(t) = \frac{1}{4} \Re \left( (e^t(I - iJ) + e^{-it}(I + iJ))(5(I - iJ) + (1 - 2i)e^{2it}(I + iJ)) \right) \bigg|_{-2i}^{1}
\]
\[
= \frac{1}{4} \Re \left( (5e^t(I - iJ)^2 + (1 - 2i)e^{it}(I + iJ)^2) \right) \bigg|_{-2i}^{1}
\]
\[
= \frac{1}{4} \Re \left( (10i - 10iJ + (1 - 2i)(2I + 2iJ)) \right) \bigg|_{-2i}^{1}
\]
\[
= \Re \left( ((3 - i)I + (1 - 2i)J) \right) \bigg|_{-2i}^{1} \rightarrow \Re \left( e^t \right) \bigg|_{-2i}^{1}
\]
\[
= \Re \left( e^t \right) \bigg|_{-2i}^{1} = \begin{bmatrix} -\cos(t) + 3\sin(t) \\ -3\cos(t) + 4\sin(t) \end{bmatrix}.
\]

The general solution of the given differential system is then:

\[
X = X_p + G(t)C.
\]

Putting \( t = 0 \), we get:

\[
C = IC = G(0)C = X(0) - X_p(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix},
\]
\[
X = \begin{bmatrix} -\cos(t) + 3\sin(t) \\ -3\cos(t) + 4\sin(t) \end{bmatrix} + e^{-t} \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}
\]
\[
= \begin{bmatrix} -\cos(t) + 3\sin(t) \\ -3\cos(t) + 4\sin(t) \end{bmatrix} + 2e^{-t} \begin{bmatrix} 2\cos(t) + \sin(t) \\ -2\sin(t) + \cos(t) \end{bmatrix}.
\]

For large positive \( t \) this asymptotically approaches the ellipse with equation:

\[
5x^2 - 6xy + 2y^2 - 5 = 0.
\]

This is an ellipse with center the origin.

It has semi-axes \( \frac{1}{2}(3\sqrt{5} \pm 5) \), with the major axis inclined to the \( x \)-axis at an angle \( \theta \), such that \( \tan(\theta) \) is the golden ratio, \( \frac{1 + \sqrt{5}}{2} \).

For large negative \( t \), the solution spirals to infinity.
A somewhat simpler method of solving this problem is to multiply the equation \( X' = AX + F \) on both sides by the operator \( E = A + (1 + i)I \).

Put \( Y = E X = (A + I)X + iX \), so \( X = \Im(Y) \).

Then we have:

\[
Y' = E X' = E(AX + F) = EAX + EF = AEX + EF = AY + L, \quad L = EF.
\]

Now we have explicitly:

\[
L = AF + (1 + i)F = 5 \begin{vmatrix} -\cos(t) + 2 \sin(t) & (1 + i) \cos(t) \\ -\cos(t) - 2 \sin(t) & (2 + i) \sin(t) \end{vmatrix} + 5 \begin{vmatrix} i \cos(t) + 2 \sin(t) & i \\ -\cos(t) + 2i \sin(t) & -1 \end{vmatrix} = 5 \begin{vmatrix} \cos(t) - 2i \sin(t) & i \\ -1 & -1 \end{vmatrix}.
\]

Put:

\[
V = \begin{vmatrix} i \\ -1 \end{vmatrix}.
\]

Note that we have:

\[
AV = \begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix} \begin{vmatrix} i \\ -1 \end{vmatrix} = \begin{vmatrix} -1 - i \\ 1 - i \end{vmatrix} = -(1 - i) \begin{vmatrix} i \\ -1 \end{vmatrix} = -(1 - i)V.
\]

So the equation \( Y' = AY + L \) is solved by:

\[
Y = y(t)V,
\]

\[
y' - Ay = y'V + (1 - i)yV = L = 5 \begin{vmatrix} \cos(t) - 2i \sin(t) & i \\ -1 & -1 \end{vmatrix} = \frac{5}{2} \begin{vmatrix} e^t + e^{-it} - 2(e^t - e^{-it}) & e^t + e^{-it} - 2(e^t - e^{-it}) \end{vmatrix} = \frac{5}{2}(-e^t + 3e^{-it}).
\]

Since this is non-resonant, a particular solution is:

\[
y = ae^{it} + be^{-it},
\]

\[
y' + (1 - i)y = iae^{it} - ibe^{-it} + (1 - i)ae^{it} + (1 - i)be^{-it} = ae^{it} + (1 - 2i)be^{-it} = -\frac{5}{2}e^{it} + \frac{15}{2}e^{-it},
\]

\[
a = -\frac{5}{2}, \quad (1 - 2i)b = \frac{15}{2}, \quad b = \frac{15}{2(1 - 2i)} = \frac{3(1 + 2i)}{2}.
\]
So the general solution for $y$ is:

$$y = ce^{-(1-i)t} + \frac{1}{2}(-5e^{it} + (3 + 6i)e^{-it}).$$

Note that we have:

$$Y(0) = EX(0) = \begin{vmatrix} i & 1 & 3 \\ -1 & i & -1 \end{vmatrix} = \begin{vmatrix} -1 + 3i \\ -3 - i \end{vmatrix} = (3+i)\begin{vmatrix} i \\ -1 \end{vmatrix} = (3+i)V = y(0)V.
$$

So $y(0) = 3 + i = c - 1 + 3i$, so $c = 4 - 2i$ and we have:

$$y = (4 - 2i)e^{-(1-i)t} + \frac{1}{2}(-5e^{it} + (3 + 6i)e^{-it})$$

$$= \cos(t)(-1 + 3i + (4-2i)e^{-t}) + \sin(t)(3 - 4i + (2+4i)e^{-t}),$$

$$\Re(y) = -\cos(t) + 3 \sin(t) + 2e^{-t}(2 \cos(t) + \sin(t)),$$

$$\Im(y) = 3 \cos(t) - 4 \sin(t) + 2e^{-t}(2 \sin(t) - \cos(t)).$$

So we obtain the required solution as:

$$X = \Im(Y) = \Im(y(t)V) = \Im \left( \begin{vmatrix} iy \\ -y \end{vmatrix} \right) = \begin{vmatrix} \Re(y) \\ -\Im(y) \end{vmatrix}$$

$$= \begin{vmatrix} -\cos(t) + 3 \sin(t) \\ -3 \cos(t) + 4 \sin(t) \end{vmatrix} + 2e^{-t} \begin{vmatrix} 2 \cos(t) + \sin(t) \\ -2 \sin(t) + \cos(t) \end{vmatrix}.$$  

This agrees with the solution found above.
Homework 8, Question 5

Find the fundamental solution $G(t) = e^{At}$ to the system: $X' = AX$, where $A$ is the following matrix:

$$A = \begin{pmatrix} -2 & -5 \\ 1 & -4 \end{pmatrix}.$$  

Use it to find the general solution, by variation of parameters of the equation $X' = AX + F$, where $F$ is the following matrix:

$$F = e^{-3t} \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}.$$  

We have:

$$\det(A) = 13, \quad \text{tr}(A) = -6, \quad A^2 + 6A + 13I = 0, \quad (A + 3I)^2 = -4I,$$

Put $J = A + 3I$. Then we have:

$$J = A + 3I = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix},$$

$$J^2 = -4I, \quad M(t) = e^{Jt} = \cos(2t)I + \frac{J}{2} \sin(2t).$$

Check:

$$M(0) = I,$$

$$M'(t) - JM = -2\sin(2t)I + J \cos(2t) - J(\cos(2t)I + \frac{J}{2} \sin(2t))$$

$$= -2\sin(2t)I - \frac{J^2}{2} \sin(2t) = 0.$$  

Then we have:

$$G(t) = e^{At} = e^{-3t}e^{(A+3I)t} = e^{-3t}e^{Jt} = \frac{1}{2}e^{-3t}(2\cos(2t)I + J \sin(2t))$$

$$= \frac{1}{2}e^{-3t} \begin{pmatrix} 2\cos(2t) + \sin(2t) & -5\sin(2t) \\ \sin(2t) & 2\cos(2t) - \sin(2t) \end{pmatrix}.$$  

Then the solution of $X' = AX + F$ is given by:

$$X = G(t)K(t),$$

$$K'(t) = G^{-1}(t)F = G(-t)F = \frac{1}{2}e^{3t}(2\cos(2t)I - J \sin(2t))e^{-3t} \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} - \frac{1}{2}(2\cos(2t)I - J \sin(2t)) \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}.$$
This is the real part of the equation:

\[
K' = \frac{1}{4}((2e^{2it} + 2e^{-2it})I + iJ(e^{2it} - e^{-2it})\begin{vmatrix} e^{it} \\ ie^a \end{vmatrix} = L \begin{vmatrix} 1 \\ i \end{vmatrix}
\]

\[
L = \frac{1}{4}(e^{3it}(2I + iJ) + e^{-it}(2I - iJ)) = \frac{1}{12i} \frac{dM}{dt},
\]

\[
M = e^{3it}(2I + iJ) - 3e^{-it}(2I - iJ).
\]

Integrating, ignoring constants of integration, we get:

\[
12iK = M \begin{vmatrix} 1 \\ i \end{vmatrix} \quad X = G(t)K(t) = \frac{1}{12i}GM \begin{vmatrix} 1 \\ i \end{vmatrix}
\]

Now we have:

\[
e^{3t}GM = \frac{1}{4}(4 \cos(2t)I + 2J \sin(2t))(e^{3it}(2I + iJ) - 3e^{-it}(2I - iJ))
\]

\[
= \frac{1}{4}((2e^{2it} + 2e^{-2it})I - iJ(e^{2it} - e^{-2it}) (e^{3it}(2I + iJ) - 3e^{-it}(2I - iJ))
\]

\[
= \frac{1}{4}(e^{2it}(2I - iJ) + e^{-2it}(2I + iJ))(e^{3it}(2I + iJ) - 3e^{-it}(2I - iJ))
\]

\[
= \frac{1}{4}((e^{5it} - 3e^{-3it})(4I + J^2) + e^{it}(2I + iJ)^2 - 3e^{it}(2I - iJ)^2)
\]

\[
= \frac{1}{4}(e^{it}(8I + 4iJ - 3(8I - 4iJ)) = 4e^{it}(-I + iJ)
\]

\[
3e^{3t}X = \frac{1}{4} \Re \begin{vmatrix} -ie^{3t}GM \\ 1 \end{vmatrix} = \Re \begin{vmatrix} e^{it}(J + iI) \\ 1 \end{vmatrix}
\]

\[
= \Re \begin{vmatrix} e^{it} \\ 1 + i \\ 1 \end{vmatrix} = \Re \begin{vmatrix} e^{it} \\ 1 - 4i \\ -i \end{vmatrix} \begin{vmatrix} 1 \\ 4 \sin(t) \\ -i \end{vmatrix}.
\]

So the required solution is:

\[
X = \frac{1}{3}e^{-3t} \begin{vmatrix} \cos(t) + 4 \sin(t) \\ \sin(t) \end{vmatrix} + G(t)C.
\]

Here C is a constant 2 \times 1 real matrix.
We check by using a simpler approach to get a particular solution. We guess:

\[ X = e^{-3t} \begin{pmatrix} a \cos(t) + b \sin(t) \\ c \cos(t) + d \sin(t) \end{pmatrix}, \]

\[ X' = e^{-3t} \begin{pmatrix} (b - 3a) \cos(t) - (a + 3b) \sin(t) \\ (d - 3c) \cos(t) - (c + 3d) \sin(t) \end{pmatrix}, \]

\[ AX = e^{-3t} \begin{pmatrix} (b - a + 5c) \cos(t) - (2a + 5c) \sin(t) \\ (d - a + c) \cos(t) + (2b + 5d) \sin(t) \end{pmatrix}, \]

\[ e^{3t}(X' - AX) = \begin{vmatrix} (b - a + 5c) \cos(t) + (5d - a - b) \sin(t) \\ (d - a + c) \cos(t) + (d - b - c) \sin(t) \end{vmatrix} = e^{3t}F = \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}. \]

Here \( a, b, c \) and \( d \) are constants to be determined.

Comparing, we need:

\[ b - a + 5c = 1, \quad 5d - a - b = 0, \quad d - a + c = 0, \quad d - b - c = -1. \]

The third equation gives \( a = c + d \).

Putting this into the other three equations gives:

\[ b - d + 4c = 1, \]

\[ 4d - c - b = 0, \]

\[ d - b - c = -1. \]

Adding the first and third of these equations gives:

\[ 3c = 0, \quad c = 0. \]

The equations now reduce to:

\[ b - d = 1, \quad 4d - b = 0. \]

Adding these two equations gives \( 3d = 1 \), so \( d = \frac{1}{3} \).

Back substituting, we get:

\[ b = 4d = \frac{4}{3}, \quad a = c + d = \frac{1}{3}, \]

\( (a, b, c, d) = \frac{1}{3}(1, 4, 0, 1). \)

So a particular solution is:

\[ X = \frac{1}{3}e^{-3t} \begin{pmatrix} \cos(t) + 4\sin(t) \\ \sin(t) \end{pmatrix}. \]

This exactly agrees with our earlier solution.
A somewhat simpler method of solving this problem is to multiply the equation \( X' = AX + F \) on both sides by the operator \( E = A + (3 + 2i)I \).

Put \( Y = EX = (A + 3I)X + 2iX \), so \( X = \frac{1}{2} \Re(Y) \).

Then we have:

\[
Y' = EX' = E(AX + F) = EAX + EF = AEX + EF = AY + L, \quad L = EF.
\]

Now we have explicitly:

\[
L = AF + (3 + 2i)F = e^{-3t} \begin{vmatrix} -2 \cos(t) + 5 \sin(t) \\ \cos(t) + 4 \sin(t) \end{vmatrix} + e^{-3t} \begin{vmatrix} (3 + 2i) \cos(t) \\ -(3 + 2i) \sin(t) \end{vmatrix} = e^{-3t} \begin{vmatrix} (1 + 2i) \cos(t) + 5 \sin(t) \\ \cos(t) + (1 - 2i) \sin(t) \end{vmatrix} = e^{-3t} (\cos(t) + (1 - 2i) \sin(t)) \begin{vmatrix} 1 + 2i \\ 1 \end{vmatrix}.
\]

Put:

\[
V = \begin{vmatrix} 1 + 2i \\ 1 \end{vmatrix}.
\]

Note that we have:

\[
AV = \begin{vmatrix} -2 & -5 \\ 1 & -4 \end{vmatrix} \begin{vmatrix} 1 + 2i \\ 1 \end{vmatrix} = \begin{vmatrix} -7 - 4i \\ -3 + 2i \end{vmatrix} = -(3 - 2i) \begin{vmatrix} 1 + 2i \\ 1 \end{vmatrix} = -(3 - 2i)V.
\]

So the equation \( Y' = AY + L \) is solved by:

\[
Y = y(t) V,
\]

\[
Y' - AY = y' V + (3 - 2i) y V = L = e^{-3t} (\cos(t) + (1 - 2i) \sin(t)) V
\]

\[
y' + (3 - 2i)y = e^{-3t} (\cos(t) + (1 - 2i) \sin(t)).
\]

Since this is non-resonant, a particular solution is:

\[
y = e^{-3t} (a \cos(t) + b \sin(t)),
\]

\[
e^{-3t} (-3a \cos(t) - 3b \sin(t) - a \sin(t) + b \cos(t) + (3 - 2i)a \cos(t) + (3 - 2i)b \sin(t))
\]

\[
= e^{-3t} (\cos(t) + (1 - 2i) \sin(t)),
\]

\[
1 = -3a + b + (3 - 2i)a = b - 2ia,
\]

\[
1 - 2i = -3b - a + (3 - 2i)b = -2ib - a,
\]

\[
2i + 1 - 2i = 1 = 2i(b - 2ia) + (-2ib - a) = 3a,
\]

\[
a = \frac{1}{3}, \quad b = 2ia + 1 = \frac{1}{3} (3 + 2i).
\]
So the general solution for $y$ is:

$$y = (p + iq)e^{-(3-2i)t} + \frac{e^{-3t}}{3}(\cos(t) + (3 + 2i)\sin(t)),$$

$$\Re(y) = \frac{1}{3}e^{-3t}(3p\cos(2t) - 3q\sin(2t) + \cos(t) + 3\sin(t)),$$

$$\Im(y) = \frac{1}{3}e^{-3t}(3p\sin(2t) + 3q\cos(2t) + 2\sin(t)).$$

Here $p$ and $q$ are real constants.

So we get:

$$X = \frac{1}{2} \Im(Y) = \frac{1}{2} \Im(y(t)V) = \frac{1}{2} \Im \left( \begin{bmatrix} y(1 + 2i) \\ y \end{bmatrix} \right)$$

$$= \frac{1}{2} \begin{vmatrix} \Im(y) + 2\Re(y) \\ \Im(y) \end{vmatrix} = \frac{1}{3}e^{-3t} \begin{vmatrix} \cos(t) + 4\sin(t) \\ \sin(t) \end{vmatrix} + \frac{1}{2}e^{-3t} \begin{vmatrix} 2p + q \cos(2t) + (p - 2q)\sin(2t) \\ p\sin(2t) + q\cos(2t) \end{vmatrix}$$

$$= \frac{1}{3}e^{-3t} \begin{vmatrix} \cos(t) + 4\sin(t) \\ \sin(t) \end{vmatrix} + \frac{1}{2}e^{-3t} \begin{vmatrix} 2\cos(2t) + \sin(2t) \\ \cos(2t) \end{vmatrix} \begin{vmatrix} p \\ q \end{vmatrix}.$$

This agrees with the solution found above. Note that if we put:

$$H(t) = \frac{1}{2}e^{-3t} \begin{vmatrix} 2\cos(2t) + \sin(2t) & \cos(2t) - 2\sin(2t) \\ \sin(2t) & \cos(2t) \end{vmatrix}.$$

Then we have:

$$H(0) = \frac{1}{2} \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}, \quad H(0)^{-1} = \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix},$$

$$H(t)H(0)^{-1} = \frac{1}{2}e^{-3t} \begin{vmatrix} 2\cos(2t) + \sin(2t) & \cos(2t) - 2\sin(2t) \\ \sin(2t) & \cos(2t) \end{vmatrix} \begin{vmatrix} 1 \\ 0 \end{vmatrix} \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix}$$

$$= \frac{1}{2}e^{-3t} \begin{vmatrix} 2\cos(2t) + \sin(2t) & -5\sin(2t) \\ \sin(2t) & 2\cos(2t) - \sin(2t) \end{vmatrix} = G(t).$$

So $H(t)$ is a fundamental solution for the homogeneous system, as expected.