Differential Equations Spring 2010
Homework Assignments

Homework 5, due Thursday 25th February 2010

Prepare for Exam 1 Thursday 25th February: topics chapters 1 through 4.
Do the following problems to be handed in for grading.

Question 1

Find the general solution of the differential equation $y''+y = 6\cos(t)+(t+1)^2$.

Question 2

Find the general solution of the differential equation $y'' + 10y' + 9y = 8e^{-t}$.

Question 3

Find the general solution of the differential equation $y'' + 8y' + 16y = 30e^{-4t}$.

Question 4

A mass of 2 kilograms dangles from a spring, stretching the spring 784 centimeters when in equilibrium.
The mass oscillates vertically in a fluid, with a frictional force 4 times its speed.
If initially it starts from rest at an extension below the equilibrium point of 100 centimeters, describe the subsequent motion, with a plot of the extension against time and with a phase space plot (use $g = 9.8$ m/s/s).

Question 5

A circuit contains an inductance of 10 henrys and a capacitance of 0.001 farads. The circuit has a driving electromagnetic force of $750\sin(5t)$ volts.
Given that the initial current and charge in the circuit is zero discuss the subsequent behavior of the current and charge in the circuit.

Question 6

Find the solution of the following differential equation and discuss the behavior of the solution as a function of t, including a phase space plot of the solution:

$$y'' + 4y = 36\sin(t), \quad y(0) = 3, \quad y'(0) = 20.$$

Also describe a physical model that could lead to this differential equation.

Homework 6, due Thursday 4th March 2010

Question 1

A circuit contains an inductance of 10 henrys, a capacitance of 0.01 farads and a resistance of 60 ohms and is driven by a voltage of $300\cos(2t)$ volts. Given that the initial charge in the system is -3 Coulombs and the initial current is 11 amperes, discuss the subsequent behavior of the current in the circuit.

Question 2

Solve the following differential equation and discuss the behavior of the solution as a function of time:

$$y'' + 4y' + 5y = -4e^{-2t}, \quad y(0) = -2, y'(0) = 5.$$

Question 3

Solve the following differential equation and discuss the behavior of the solution as a function of time, including a phase space plot of the solution:

$$y'' + 5y' + 4y = 10\sin(2t) + 20\cos(2t), \quad y(0) = 2, y'(0) = 1.$$

Also describe a physical model for this differential equation.

Question 4

By calculating a suitable Wronskian, find, with proof, the general solution of the differential equation $y'' - 4y' + 4y = 0$.

Question 5

Using the technique of variation of parameters, solve the following differential equation

$$y'' - 9y = 12e^{-3t}, \quad y(0) = 4, \ y'(0) = -8.$$

Also discuss the behavior of the solution, with graphs of y against time and with a phase space plot.

Question 6

Using the technique of variation of parameters, solve the following differential equation

$$t^2y'' + 7ty' - 16y = 24t^4, \quad y(1) = 4,\ y'(1) = -10.$$

Also discuss the behavior of the solution, with graphs of y against time and with a phase space plot.

Homework 7, due Thursday 25th March 2010

Question 1

Use Euler's method with step size $h = 0.1$ to compute the first five iterations for the differential equation:

$$y' = t + y, \quad y(0) = 1.$$

Plot your results.
Also determine and plot the exact solution and compare your results with the exact solution.

Question 2

Solve the matrix equations $AXA^{-1} = B$ and $A^{-1}YA = B$, for the unknown 2×2-matrices X and Y, where A and B are the following matrices:

$$A = \begin{vmatrix} 5 & -6 \\ 7 & -9 \end{vmatrix}, \qquad B = \begin{vmatrix} 2 & 3 \\ -3 & -5 \end{vmatrix}.$$

Also show that the solution of the equation $AZA^{-1} = B^3$ is $Z = X^3$.

Question 3

Solve the following system:

$$\frac{d}{dt}\begin{vmatrix} x \\ v \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -2 & 2 \end{vmatrix}\begin{vmatrix} x \\ v \end{vmatrix}.$$

(Hint: write out the system, show that v can be replaced in it by x' and therefore v' by x'' and then solve the resulting second order equation for x).
The initial conditions are $x(0) = -1$, $v(0) = 0$.
Plot the solution and describe the plot.

Question 4

Find the general solution of the following system and discuss the resulting trajectories:

$$x' = -x^3y^2, \quad y' = -\frac{1}{2}(x-2)y, \quad x(0) = \frac{1}{2}, \quad y(0) = 2.$$

(Hint: first obtain an equation for y in terms of x).

Question 5

Tanks A and B each contain two hundred liters of brine.
Initially tank A contains fifty kilograms of salt, whereas tank B contains only pure water.

- Pure water enters tank A from a tap at a rate of five liters per minute.
- Six liters per minute of well-mixed fluid flows out from tank A along a pipe into tank B.
- The well-mixed fluid in tank B flows along a second pipe back into tank A at a rate of one liter per minute.
- The well-mixed fluid is also pumped out of tank B into a reservoir at a rate of five liters per minute.

Determine the amount of salt in each tank as a function of time and discuss your results.
If the reservoir is initially empty, also determine the amount of salt in the reservoir as a function of time and discuss your results.

Question 6

Solve the system: $X' = AX + F$, with initial condition $X(0)$, given as follows:

$$A = \begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix}, \quad F = \begin{vmatrix} 5\cos(t) \\ 10\sin(t) \end{vmatrix}, \quad X(0) = \begin{vmatrix} 3 \\ -1 \end{vmatrix}.$$

Plot the solution and discuss its behavior.

Homework 8, due Thursday 1st April 2010

Question 1

Classify the phase portraits (sink, source, spiral, node, saddle, etc.) for the system $X' = AX$, for the following cases:

- $A = \begin{vmatrix} 3 & 2 \\ -1 & -1 \end{vmatrix}$
- $A = \begin{vmatrix} -4 & 4 \\ -5 & 2 \end{vmatrix}$
- $A = \begin{vmatrix} 1 & 8 \\ 2 & 1 \end{vmatrix}$

Question 2

Set up Euler's method for the system $x' = -y$, $y' = 4x$, with initial condition $(x, y)(0) = (2, 1)$ and arbitrary step size h as a matrix recursion.
Compute the time 1 flow with $h = \frac{1}{4}$, $h = 0.1$ and $h = 0.05$ and plot your results.
Also compare with the exact solution.
Discuss your results.

Question 3

Let A be the following matrix:

$$A = \begin{vmatrix} 1 & -3 \\ -4 & 0 \end{vmatrix}$$

Compute the fundamental solution $G(t) = e^{At}$ and use it to solve by variation of parameters the equation:

$$X' = AX + F, \quad F = \begin{vmatrix} 3 \\ 7e^{-3t} \end{vmatrix}, \quad X(0) = \begin{vmatrix} 4 \\ -2 \end{vmatrix}.$$

Discuss the behavior of the solution, both forward and backward in time.

Question 4

Solve the matrix system $X' = AX$, with initial condition $X(0)$ given as follows:

$$A = \begin{vmatrix} -3 & -1 \\ 1 & -1 \end{vmatrix}, \quad X(0) = \begin{vmatrix} 4 & -2 \\ 1 & -3 \end{vmatrix}.$$

Plot the solution and discuss its behavior.

Question 5

Find the fundamental solution $G(t) = e^{At}$ to the system: $X' = AX$, where A is the following matrix:

$$A = \begin{vmatrix} -2 & -5 \\ 1 & -4 \end{vmatrix}.$$

Use it to find the general solution, by variation of parameters of the equation $X' = AX + F$, where F is the following matrix:

$$F = e^{-3t} \begin{vmatrix} \cos(t) \\ -\sin(t) \end{vmatrix}.$$

Question 6

Consider the following predator-prey system:

$$10x' = 12x - xy,$$

$$10y' = -6y + xy.$$

- Find the equilibrium solutions.
- Plot the phase plane, with several solutions superimposed.
- Discuss the qualitative behavior of the solution with the initial condition: $(x(0), y(0)) = (1, 3)$.
- Find and plot the level surfaces of the solution: hint $\dfrac{dy}{dx} = \dfrac{y'}{x'}$ gives a separable differential equation, which can be solved.
  In particular plot the level surface passing through the point $(1, 3)$.
- Discuss the stability of the equilibrium solutions.

Homework 9, due Thursday 15th April 2010

Question 1

Consider the following predator-prey system:

$$10x' = 2x - xy,$$

$$10y' = -3y + xy.$$

- Which is the predator and which the prey? Explain your answer.
- Find the equilibrium solutions.
- Plot the phase plane, with several solutions superimposed.
- Discuss the qualitative behavior of the solution with the initial condition: $(x(0), y(0)) = (3, 2)$.
- Find and plot the level surfaces of the solution.
  In particular plot the level surface passing through the point $(3, 2)$.
- Discuss the stability of the equilibrium solutions.

Question 2

Consider the following predator-prey system:

$$1000x' = 400x - x^2 - 10xy,$$

$$1000y' = -300y + 5xy.$$

- Which is the predator and which the prey? Explain your answer.
- Find the equilibrium solutions.
- Plot the phase plane, with several solutions superimposed.
- Discuss the qualitative behavior of the solution with the initial condition: $(x(0), y(0)) = (60, 34)$.
- Discuss the stability of the equilibrium solutions.

Question 3

Find the matrix exponential function for each of the following matrices:

- $A = \begin{vmatrix} -2 & 1 \\ -1 & 0 \end{vmatrix}$
- $B = \begin{vmatrix} -2 & 6 \\ -1 & 5 \end{vmatrix}$
- $C = \begin{vmatrix} -2 & 4 \\ -1 & -2 \end{vmatrix}$

Question 4

Consider the following predator-prey system:

$$x' = 400x - 5x^2 - 10xy,$$
$$y' = -500y + 10xy.$$

- Which is the predator and which the prey? Explain your answer.
- Find the equilibrium solutions.
- Perform a stability analysis for each equilibrium solution.
- Discuss the fate of the system for the initial condition $(x, y) = (40, 10)$.

Question 5

Consider the following system representing the population of two species competing for resources:

$$x' = (10 - 2x - y)x,$$
$$y' = (15 - x - 3y)y.$$

- Find the equilibrium solutions.
- Perform a stability analysis for each equilibrium solution.
- Plot the direction field for the system.
- Discuss the behavior of the system for the initial condition $(x, y) = (2, 2)$.

Question 6

Let a circuit have a resistance 10 ohms, a capacitance of 0.125 Farads and an inductance of 40 henrys, all connected in parallel. Let the voltage drop across the capacitance be V volts and the current through the inductor be J amperes.

Show that the voltage and current obey the equations:

$$V' = -\frac{4V}{5} - 8J,$$

$$J' = \frac{V}{40}.$$

Write this system as a first order matrix differential system and find its general solution.
Given that the initial current is 10 amps and the initial voltage is 30 volts, find the solution and discuss its behavior.

Homework 10, due Thursday 22nd April 2010

Question 1

Let a circuit have in parallel:

- A resistance 10 ohm.
- A capacitance 0.125 farad
- A resistance of 5 ohms and an inductance of 40 henry (connected to each other in series).

Let the voltage drop across the capacitance be V volts and the current through the inductor be J amperes.

Model the system as a first order matrix differential system and find its general solution.
Given that the initial current is 20 amps and the initial voltage is 100 volts, find the solution and discuss its behavior.

Question 2

Consider the following differential system:

$$x' = (6 - 2x - 3y)x,$$

$$y' = (1 - x - y)y.$$

- Find the equilibrium solutions.
- Perform a stability analysis for each equilibrium solution.
- Plot the direction field for the system.
- Discuss the behavior of the system for the initial condition $(x, y) = (1, 3)$.

Question 3

Consider the following differential system:

$$x' = \sin(x)y,$$

$$y' = \cos(x) - y.$$

- Find the equilibrium solutions.
- Perform a stability analysis for each equilibrium solution.
- Plot the direction field for the system.
- Discuss the behavior of the system for the initial condition $(x, y) = (4, 4)$.

Question 4

Determine if the following systems are Hamiltonian. If one is find the Hamiltonian and plot its level surfaces. In the cases that the system is Hamiltonian, use your results to discuss the behavior of the system.

- $x' = 2y + x, y' = -2x + x^3$
- $x' = 5x - y^2 + 2xy, y' = 3x^2 - 5y - y^2$
- $x' = y, y' = -2x + x^3$.

Question 5

Find the general series expansion for the solutions of the equation $y'' = 4y$. Also solve the equation exactly and verify that the series you have obtained correspond to the exact solutions.

Question 6

Find, to order four, the power series expansions, valid near $x = 0$, for the two linearly independent solutions for the differential equation:

$$(1 + x^2)y'' + y' - 2y = 0.$$