Question 1
Consider the linear system in the variables \((x, y, z, t, u)\), given by the following matrix, in echelon form:

\[
\begin{array}{ccccc|c}
1 & 2 & -1 & 3 & 1 & 2 \\
0 & 1 & 1 & 3 & -1 & 4 \\
0 & 0 & 1 & 2 & 3 & 3
\end{array}
\]

- Reduce the system to reduced echelon form and give the general solution.
- Write your answer in the form of the sum of a particular solution and the general solution of the associated homogeneous system.
- How many free variables are there?
- Which are the pivot variables?
- Is there a solution with \(y = z = t = 1\) and if so is what is the solution and is it unique?
Question 2

Consider the following linear system:

\[
\begin{align*}
    x - 3y + 2z &= 8 \\
    3x - 8y - 5z &= 11 \\
    2x - 4y - 18z &= -10
\end{align*}
\]

• Find the general solution of the system.
• Give a geometric interpretation of the system and your solution.
• Is there a solution of the system with \(x = 0\)?
  If so, find it.
Question 3

Everyplace.com has three levels of employee, levels A, B and C.

- Last year level A employees each received 10,000 stock options, level B employees each received 5,000 stock options and level C employees 2,500 stock options.

- Bonuses for a record year were paid out at $20,000 for levels A and B and $10,000 for level C.

- Base salaries were $120,000 for level A, $80,000 for level B and $50,000 for level C.

- Last year a total of 300,000 stock options were given out, total bonuses of $1,000,000 and total base salaries of $5,000,000.

How many employees does Everyplace.com have?
Question 4

Give an example of each of the following or explain why no such example can exist:

- An inconsistent linear system in three variables, with a coefficient matrix of rank two.
- A consistent linear system with three equations and two unknowns, with a coefficient matrix of rank one.
- A consistent linear system with three equations and two unknowns, with a coefficient matrix of rank larger than one.
- A linear system of two equations in three unknowns, with an invertible coefficient matrix.
- A linear system in three variables, whose geometrical interpretation is three planes intersecting in a line.
Question 5

Let $A$ be the following matrix:

$$
\begin{bmatrix}
1 & 3 \\
-2 & -8
\end{bmatrix}
$$

- Compute the matrices $A^2$, $AA^T$ and $A^{-1}$.
- Find numbers $p$ and $q$, such that $A^2 = pA + qI$, where $I$ is the $2 \times 2$ identity matrix.
- Write $A$ and $A^T$ as a product of elementary matrices.
- Let $B = A - tI$, where $t$ is a scalar.
  For which values of $t$ is $B$ not invertible?
Question 6

Calculate the following determinant (using suitable properties of the determinant to simplify the calculation):

\[
\begin{vmatrix}
  x & y & z & 1 \\
  1 & -2 & 3 & 1 \\
  2 & -3 & 1 & 1 \\
  4 & -6 & 3 & 1 \\
\end{vmatrix}
\]

Also give the geometrical interpretation of the vanishing of the determinant.
Question 7

Let $A$ and $B$ be the following matrices:

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 3 & 1 & 0 \\ 2 & 3 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 3 & -1 \end{bmatrix}$$

By row reducing a suitable matrix, solve the equation $AX = B$.

Is the matrix $A$ invertible?

Explain your answer.