Theoretical Mathematics II, Quiz 4 Solutions, 11/12/9

Question 1

Let \( f : [0, 1] \to \mathbb{R} \) be continuous, with \( f(0) = 1 \) and \( f(1) = 0 \).

Suppose that \( f \) is injective.

Show that \( f \) has range \([0, 1]\).

Suppose \( y \notin [0, 1] \) lies in the range of \( f \), which is the set \( f([0, 1]) \).

Then either \( y > 1 \), or \( y < 0 \).

- Suppose that \( y > 1 \).
  
  Let \( y = f(a) \), with \( a \in [0, 1] \).
  
  Then \( a \neq 0 \), since \( f(0) = 1 \neq y \) and \( a \neq 1 \), since \( f(1) = 0 \neq y \).
  
  So \( a \in (0, 1) \). So \( 0 < a < 1 \).

  Now consider the continuous function \( f \) on the closed interval \([a, 1]\).

  We have \( f(a) = y > 1 \) and \( f(1) = 0 < 1 \).

  By the intermediate value theorem there is a number \( z \in (a, 1) \), such that \( f(z) = 1 \).

  Then \( z > a > 0 \), so \( z \neq 0 \). But \( f(z) = 1 = f(0) \).

  So \( f \) is not injective, a contradiction.

- Suppose that \( y < 0 \).
  
  Let \( y = f(b) \), with \( b \in [0, 1] \).
  
  Then \( b \neq 0 \), since \( f(0) = 1 \neq y \) and \( b \neq 1 \), since \( f(1) = 0 \neq y \).
  
  So \( b \in (0, 1) \). So \( 0 < b < 1 \).

  Now consider the continuous function \( f \) on the closed interval \([0, b]\).

  We have \( f(0) = 1 > 0 \) and \( f(b) = y < 0 \).

  By the intermediate value theorem there is a number \( t \in (0, b) \), such that \( f(t) = 0 \).

  Then \( t < b < 1 \), so \( t \neq 1 \). But \( f(t) = 0 = f(1) \).

  So \( f \) is not injective, a contradiction.

So the hypothesis that \( y \notin [0, 1] \) lies in the range of \( f \) leads to a contradiction.

So the range of \( f \) is a subset of \([0, 1]\).

Finally, if \( u \in [0, 1] \), we have \( f(1) = 0 \leq u \leq 1 = f(0) \), so by the intermediate value theorem again, some \( s \) exists with \( 0 \leq s \leq 1 \) and \( f(s) = u \).

So \( u \) lies in the range of \( f \). So the set \([0, 1]\) is a subset of the range of \( f \).

Since we have \( f([0, 1]) \subseteq [0, 1] \subseteq f([0, 1]) \), we get: \( f([0, 1]) = [0, 1] \), so the range of \( f \) is the closed interval \([0, 1]\), as required.
Question 2
Determine, from first principles (by computing an appropriate limit) the following derivatives, or prove that the derivative in question does not exist.

\( \bullet f = x^{-2}, \text{ find } f'(2). \)

We need the limit:

\[
f'(2) = \lim_{x \to 2} \left( \frac{f(x) - f(2)}{x - 2} \right).
\]

We have:

\[
\frac{f(x) - f(2)}{x - 2} = \frac{\frac{1}{x^2} - \frac{1}{4}}{x - 2}.
\]

Multiply the numerator and denominator by \(-4x^2\), giving:

\[
\frac{f(x) - f(2)}{x - 2} = \frac{x^2 - 4}{-4x^2(x - 2)} = \frac{(x - 2)(x + 2)}{-4x^2(x - 2)} = -\left( \frac{x + 2}{4x^2} \right).
\]

So we have:

\[
f'(2) = \lim_{x \to 2} \left( \frac{f(x) - f(2)}{x - 2} \right) = -\lim_{x \to 2} \left( \frac{x + 2}{4x^2} \right) = -\left( \frac{2 + 2}{4(2^2)} \right) = -\frac{4}{16} = -\frac{1}{4}.
\]
• \( g = \sqrt{x + 1} \), find \( g'(3) \).

We need the limit:

\[
g'(3) = \lim_{x \to 3} \left( \frac{g(x) - g(3)}{x - 3} \right)
\]

We have:

\[
\frac{g(x) - g(3)}{x - 3} = \frac{\sqrt{x + 1} - 2}{x - 3}.
\]

Multiply the numerator and denominator by \( \sqrt{x + 1} + 2 \), giving:

\[
\frac{g(x) - g(3)}{x - 3} = \frac{\sqrt{x + 1} - 2}{x - 3} \cdot \frac{\sqrt{x + 1} + 2}{\sqrt{x + 1} + 2} = \frac{(\sqrt{x + 1})^2 - 2^2}{(x - 3)(\sqrt{x + 1} + 2)} = \frac{x + 1 - 4}{(x - 3)(\sqrt{x + 1} + 2)} = \frac{x - 3}{(x - 3)(\sqrt{x + 1} + 2)} = \frac{1}{\sqrt{x + 1} + 2}.
\]

So we have:

\[
g'(3) = \lim_{x \to 3} \left( \frac{g(x) - g(3)}{x - 3} \right) = \lim_{x \to 3} \left( \frac{1}{2 + \sqrt{x + 1}} \right) = \left( \frac{1}{2 + \sqrt{3 + 1}} \right) = \frac{1}{2 + 2} = \frac{1}{4}.
\]
• \( h = |x| \sqrt{|x|} \), find \( h'(0) \).

We note that \( h(0) = 0 \).
We need:

\[
h'(0) = \lim_{x \to 0} \frac{h(x) - h(0)}{x - 0}
\]
\[
= \lim_{x \to 0} \frac{h(x)}{x}
\]
\[
= \lim_{x \to 0} j(x),
\]
\[
j(x) = \frac{|x| \sqrt{|x|}}{x}, \text{ whenever } x \neq 0.
\]

But we have, for any \( x \neq 0 \),

\[
0 < |j(x)| = \left| \frac{|x| \sqrt{|x|}}{x} \right| = \frac{|x| \sqrt{|x|}}{|x|} = \sqrt{|x|}.
\]

Now as \( x \to 0 \), we have \( |x| \to 0 \), so \( \sqrt{|x|} \to 0 \) also.
So by squeeze, we have \( |j(x)| \to 0 \), as \( x \to 0 \).
So \( -|j(x)| \to 0 \), also, as \( x \to 0 \).
But \( -|j(x)| \leq j(x) \leq |j(x)| \), so by squeeze again, we have \( j(x) \to 0 \),
as \( x \to 0 \).
So the required derivative \( h'(0) \) exists and is 0.
Question 3

Let \( f(x) = \sin \left( \frac{\pi x}{3} \right) \) and \( g(x) = \sqrt{x^2 - 9} \).

Determine, with proof, the following derivatives:

- \((f \circ f)'(3)\)
- \((g \circ g)'(5)\)
- \((f \circ g)'(5)\)

We have, by the chain rule:

\[
f'(x) = \frac{\pi}{3} \cos \left( \frac{\pi x}{3} \right), \quad g'(x) = \frac{x}{\sqrt{x^2 - 9}}.
\]

- For \((f \circ f)'(3)\), by the chain rule, we have:

\[
(f \circ f)'(3) = (f' \circ f)(3) f'(3) = f'(f(3)) f'(3).
\]

Now we have: \( f(3) = \sin(\pi) = 0 \), so we get:

\[
(f \circ f)'(3) = f'(0) f'(3) = \left( \frac{\pi}{3} \right)^2 \cos(0) \cos(\pi) = \frac{-\pi^2}{9}.
\]

- For \((g \circ g)'(5)\), by the chain rule, we have:

\[
(g \circ g)'(5) = (g' \circ g)(5) g'(5) = g'(g(5)) g'(5).
\]

Now we have: \( g(5) = \sqrt{25 - 9} = \sqrt{16} = 4 \), so we get:

\[
(g \circ g)'(5) = g'(4) g'(5) = g'(4) g'(5) = \left( \frac{4}{\sqrt{16 - 9}} \right) \left( \frac{5}{\sqrt{25 - 9}} \right) = \frac{5}{\sqrt{7}} = \frac{5\sqrt{7}}{7}.
\]

- For \((f \circ g)'(5)\), by the chain rule, we have:

\[
(f \circ g)'(5) = (f' \circ g)(5) g'(5) = f'(g(5)) g'(5).
\]

Now we have: \( g(5) = \sqrt{25 - 9} = \sqrt{16} = 4 \), so we get:

\[
(f \circ g)'(5) = f'(g(5)) g'(5) = f'(4) g'(5) = \frac{\pi}{3} \cos \left( \frac{4\pi}{3} \right) \left( \frac{5}{\sqrt{25 - 9}} \right) = \frac{\pi}{3} \left( \frac{-1}{2} \right) \left( \frac{5}{4} \right) = \frac{5\pi}{24}.
\]