Theoretical Mathematics II, Exam 2 Solutions, 11/24/9

Question 1

Let $f : [0, 1] \to [0, 1]$ be continuous.

- Prove that $f$ is injective if and only if $f$ is strictly monotonic.

First suppose that $f$ is strictly monotonic.
Let $s$ and $t$ be reals with $0 \leq s \leq 1$ and $0 \leq t \leq 1$ and $f(s) = f(t)$.

- If $f$ is strictly increasing, and $s < t$, then $f(s) < f(t)$, false; so $s \geq t$; if $s > t$, then $f(s) > f(t)$, false. So $s = t$.

- If $f$ is strictly decreasing, and $s < t$, then $f(s) > f(t)$, false; so $s \geq t$; if $s > t$, then $f(s) < f(t)$, false. So $s = t$.

In either case $f(s) = f(t)$ implies that $s = t$, so $f$ is injective.

Now suppose that $f$ is injective.
We want to show that $f$ is strictly monotonic.
Note that the function $g$, given by the formula $g(x) = 1 - f(x)$, for any $x \in [0,1]$, is also continuous and injective and $g : [0, 1] \to [0, 1]$, and $f$ is strictly monotonic if and only if $g$ is (where $f$ is strictly increasing if and only if $g$ is strictly decreasing, whereas $f$ is strictly decreasing if and only if $g$ is strictly increasing).
Also $g(0) = f(1)$ and $g(1) = f(0)$.
Since $f$ is injective, either $f(0) < f(1)$, or $f(0) > f(1)$.
If $f(1) > f(0)$, then $g(0) < g(1)$.
So at worst by replacing $f$ by $g$, we map assume, henceforth, without loss of generality that $f(0) < f(1)$.
We will show that $f$ is necessarily strictly increasing.
Let $0 < x < 1$.

- First we have: $f(x) \neq f(0)$ and $f(x) \neq f(1)$, by injectivity of $f$.
- Next if $f(x) < f(0)$, by the intermediate value theorem, applied to the interval $[x, 1]$, there is a point $y$ in the interval $(x, 1)$ with $f(y) = f(0)$, contrary to injectivity of $f$.
- Next if $f(x) > f(1)$, by the intermediate value theorem, applied to the interval $[0, x]$, there is a point $z$ in the interval $(0, x)$ with $f(z) = f(1)$, contrary to injectivity of $f$.

So $0 < x < 1$ implies that $f(0) < f(x) < f(1)$.

Now suppose that $0 < x < y < 1$.

Then we have $f(0) < f(x) < f(1)$ and $f(0) < f(y) < f(1)$ and $f(x) \neq f(y)$ (by injectivity of $f$).

If it were always the case that $f(x) < f(y)$, then $f$ would be strictly increasing on $[0, 1]$.

So if $f$ is not strictly increasing, real numbers $p$ and $q$ must exist with $0 < p < q < 1$ and $f(0) < f(q) < f(p) < f(1)$.

Then by the intermediate value theorem, applied to the interval $[0, p)$, some point $r$ exists with $0 < r < p$ and $f(r) = f(q)$, but $r < p < q$, so $r < q$, contrary to injectivity of $f$.

So $f$ is strictly increasing, so, a fortiori, strictly monotonic and we are done.
Question 2

Let \( f(x) = 2x^3 \) if \( x \) is rational, whereas \( f(x) = x^4 + x^2 \) if \( x \) is irrational.

- Determine, with proof, all points at which \( f \) is differentiable.

If \( x = c \) gives a point of differentiability, the function \( f(x) \) must be continuous at \( c \).

Let \( x_n \) be a sequence of rationals with \( x_n \to c \).
Then we need \( f(x_n) \to f(c) \), so \( 2x_n^3 \to f(c) \).
But \( 2x_n^3 \to 2c^3 \), so \( f(c) = 2c^3 \).

Let \( y_n \) be a sequence of irrationals with \( y_n \to c \).
Then we need \( f(y_n) \to f(c) \), so \( x_n^4 + x_n^2 \to f(c) \).
But \( x_n^4 + x_n^2 \to c^4 + c^2 \), so \( f(c) = c^4 + c^2 \).

So if \( c \) is a point of continuity of \( f \), we need \( 2c^3 = c^4 + c^2 \), which gives:
\[
0 = c^4 - 2c^3 + c^2 = c^2(c^2 - 2c + 1) = c^2(c - 1)^2, \quad \text{so} \quad c = 0, \quad \text{or} \quad c = 1.
\]

- When \( x = 1 \), we have \( f(1) = 2 \) and then we get:

\[
f(1) = \lim_{x \to 1} g(x),
\]

\[
g(x) = \frac{f(x) - f(1)}{x - 1} = \frac{f(x) - 2}{x - 1}, \quad x \neq 1.
\]

Put \( g_1(x) = 2(x^2 + x + 1) \) and \( g_2(x) = x^3 + x^2 + 2x + 2 \), for any real \( x \).

When \( 1 \neq x \) is rational, we have:

\[
g(x) = 2 \left( \frac{x^3 - 1}{x - 1} \right) = 2 \left( \frac{x^3 - x^2 + x^2 - x + x - 1}{x - 1} \right) = 2(x^2 + x + 1) = g_1(x).
\]

When \( x \) is irrational, we have:

\[
g(x) = \frac{x^4 + x^2 - 2}{x - 1} = \frac{x^4 - x^3 + x^3 - x^2 + 2x^2 - 2x + 2x - 2}{x - 1} = x^3 + x^2 + 2x + 2 = g_2(x).
\]

Now we have \( \lim_{x \to 1} g_1(x) = g_1(1) = 6 = g_2(1) = \lim_{x \to 1} g_2(x) \).

So for given \( \epsilon > 0 \), we can choose \( \delta(\epsilon) > 0 \), so that if \( |x - 1| < \delta(\epsilon) \),
then we have both \( |g_1(x) - 6| < \epsilon \) and \( |g_2(x) - 6| < \epsilon \).

Then we have also, if \( |x - 1| < \delta(\epsilon) \), \( |g(x) - 6| < \epsilon \).

So \( \lim_{x \to 1} g(x) = 6 \) and \( f'(1) = 6 \).
• When \( x = 0 \), we have \( f(0) = 0 \) and then we get:

\[
f'(0) = \lim_{x \to 0} h(x),
\]

\[
h(x) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}, \quad x \neq 0.
\]

Put \( h_1(x) = 2x^2 \) and \( h_2(x) = x^3 + x \), for any real \( x \).
When \( 0 \neq x \) is rational, we have:

\[
h(x) = \frac{2x^3}{x} = 2x^2 = h_1(x).
\]

When \( x \) is irrational, we have:

\[
h(x) = \frac{x^4 + x^2}{x} = x^3 + x = h_2(x).
\]

Now we have \( \lim_{x \to 0} h_1(x) = h_1(0) = 0 = h_2(0) = \lim_{x \to 0} h_2(x) \).
So for given \( \epsilon > 0 \), we can choose \( \delta(\epsilon) > 0 \), so that if \( |x| < \delta(\epsilon) \), then we have both \( |h_1(x)| < \epsilon \) and \( |h_2(x)| < \epsilon \).
Then we have also, if \( |x| < \delta(\epsilon) \), \( |h(x)| < \epsilon \).
So \( \lim_{x \to 0} h(x) = 0 \) and \( f'(0) = 0 \).

So the given function is differentiable only at the points \( x = 0 \) and \( x = 1 \), with derivatives \( f'(0) = 0 \) and \( f'(1) = 6 \).
**Question 3**

Determine from first principles (by computing an appropriate limit), the following derivatives, or prove that the derivative in question does not exist:

- $f(x) = \frac{1}{\sqrt{x}}$.

  Find $f'(9)$.

  We have $f(9) = \frac{1}{\sqrt{9}} = \frac{1}{3}$ and $x - 9 = (\sqrt{x} - 3)(\sqrt{x} + 3)$, for any $x > 0$, so we get:

  $$f'(9) = \lim_{x \to 9} \left( \frac{f(x) - f(9)}{x - 9} \right)$$

  $$= \lim_{x \to 9} \left( \frac{\frac{1}{\sqrt{x}} - \frac{1}{3}}{x - 9} \right) = \lim_{x \to 9} \frac{3 - \sqrt{x}}{(\sqrt{x} - 3)(\sqrt{x} + 3)3\sqrt{x}}$$

  $$= -\lim_{x \to 9} \frac{1}{(\sqrt{x} + 3)3\sqrt{x}} = -\frac{1}{(\sqrt{9} + 3)3\sqrt{9}} = -\frac{1}{54}.$$ 

- $g(x) = \left( x - \frac{2}{x} \right)^2$.

  Find $g'(1)$.

  We have $g(1) = (-1)^2 = 1$, so we get:

  $$g'(1) = \lim_{x \to 1} \left( \frac{g(x) - 1}{x - 1} \right) = \lim_{x \to 1} \frac{\left( x - \frac{2}{x} \right)^2 - 1}{x - 1}$$

  $$= \lim_{x \to 1} \frac{(x - 2) (x - 1)}{x - 1}$$

  $$= \lim_{x \to 1} \frac{(x^2 - x - 2) (x + 2)}{x^2(x - 1)} = \lim_{x \to 1} \frac{1}{x^2} \frac{(x^2 - x - 2) (x + 2)}{(x - 1)(x + 2)}$$

  $$= \lim_{x \to 1} \frac{(x^2 - x - 2) (x + 2)}{x^2} = \lim_{x \to 1} \frac{1 - 2}{1^2} = -6.$$
• \( h(x) = |x|^{\frac{2}{3}}. \)

Find \( h'(0) \).

We have \( h(0) = 0 \), so we get:

\[
h'(0) = \lim_{{x \to 0}} j(x),
\]

\[
j(x) = \frac{h(x) - h(0)}{x - 0} = \frac{|x|^{\frac{2}{3}}}{x}, \quad 0 \neq x.
\]

Put \( x_n = n^{-3} > 0 \), for any \( n \in \mathbb{N} \).
Then \( x_n \to 0 \), as \( n \to \infty \).
But we have:

\[
j(x_n) = \frac{|x_n|^{\frac{2}{3}}}{x_n} = x_n^{-\frac{1}{3}} = n.
\]

So \( j(x_n) \to \infty \) as \( n \to \infty \).
So \( \lim_{{x \to 0}} j(x) \) does not exist.
So the function \( h(x) \) is not differentiable at \( x = 0 \).
Question 4

Let \( f(x) = \frac{1}{3 - x^2} \) and \( g(x) = \tan \left( \frac{\pi x}{4} \right) \).

Determine, with proof, the following derivatives:

- \((f \circ f)'(2)\).
- \((g \circ f)'(2)\).
- \((f \circ g)'(1)\).

We have \( f(2) = -1 \) and \( g(1) = \tan \left( \frac{\pi}{4} \right) = 1 \).

Also we have, by the chain rule:

\[
\begin{align*}
f'(x) &= 2x(3 - x^2)^{-2}, \\
f'(-1) &= -2(2^{-2}) = -\frac{1}{2}, \\
f'(1) &= 2(2^{-2}) = \frac{1}{2}, \\
f'(2) &= 4, \\
g'(x) &= \frac{\pi}{4} \sec^2 \left( \frac{\pi x}{4} \right), \\
g'(1) &= g'(-1) = \frac{\pi}{4 \cos^2 \left( \frac{\pi}{4} \right)} = \frac{\pi}{2}.
\end{align*}
\]

Then by the chain rule, we have:

- \((f \circ f)'(2) = f'(f(2))f'(2) = f'(-1)f'(2) = -2,\)
- \((g \circ f)'(2) = g'(f(2))f'(2) = g'(-1)f'(2) = 2\pi,\)
- \((f \circ g)'(1) = f'(g(1))g'(1) = f'(1)g'(1) = \frac{\pi}{4}.\)
Question 5

Let \( f(x) \) and \( g(x) \) be differentiable functions, such that:
\( f(4) = 5, \ g(5) = 4, \ f'(4) = 2 \) and \( g'(5) = -2 \).
Determine, with proof the following derivatives:

- \( h(x) = f(x^2 - 5); \) find \( h'(3) \).

By the chain rule, we have:
\[
h'(x) = 2xf'(x^2 - 5),
\]
\[
h'(3) = 2(3)f'(3^2 - 5) = 6f'(4) = 12.
\]

- \( j(x) = f^{-2}(x) + g^2(x^2 - 11); \) find \( j'(4) \).

By the chain rule, we have:
\[
j'(x) = -2f^{-3}(x)f'(x) + 2(2x)g(x^2 - 11)g'(x^2 - 11),
\]
\[
j'(4) = -2(5^{-3})(2) + 2(8)g(5)g'(5) = -\frac{4}{125} + 16(4)(-2) = -128 - \frac{32}{1000} = -128.032.
\]

- \( k(x) = (g \circ f)(x^2); \) find \( k'(2) \).

By the chain rule, we have:
\[
k'(x) = g'(f(x^2))f'(x^2)(2x),
\]
\[
k'(2) = g'(f(4))f'(4)(4) = 4g'(5)(2) = 4(-2)(2) = -16.
\]
Question 6

Let $f(x) = x^3 - 3x + 2$.
Find, with proof, the range of $f$ on the interval $[-3, 2]$.
Also determine, with proof, all real open intervals, on which $f$ is invertible.

We first note that $f$ is everywhere continuous and everywhere differentiable.

Next we have:

$$f(-3) = -27 + 9 + 2 = -16,$$
$$f(-1) = -1 + 3 + 2 = 4,$$
$$f(1) = 1 - 3 + 2 = 0,$$
$$f(2) = 8 - 6 + 2 = 4.$$

Next, we have:

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x - 1)(x + 1).$$

- When $x \geq 1$, $f'(x) \geq 0$, so $f'(x) \geq 0$ on the interval $[1, 2]$, so $f$ is increasing on that interval so, by the intermediate value theorem, $f$ has range $[f(1), f(2)] = [0, 4]$.

- When $x \leq -1$, $f'(x) \geq 0$, so $f'(x) \geq 0$ on the interval $[-3, -1]$, so $f$ is increasing on that interval so, by the intermediate value theorem, $f$ has range $[f(-3), f(-1)] = [-16, 4]$.

- When $-1 \leq x \leq 1$, $f'(x) \leq 0$, so $f$ is decreasing on the interval $[-1, 1]$, so, by the intermediate value theorem, has range $[f(1), f(-1)] = [0, 4]$.

So the range of $f$ on the interval $[-3, 2]$ is the union of the intervals $[0, 4]$, $[0, 4]$ and $[-16, 4]$, so is the closed interval $[-16, 4]$. 

For the last part:

- \( f \) is not invertible on any open interval containing the point \( x = -1 \), since on any interval of the form \((a, b)\) with \( a < -1 < b \), we have \( f \) increasing on the interval \((a, -1]\), so it has range \((f(a), 4]\) on that interval, where \( f(a) < 4 \) and we have \( f \) decreasing on the interval \([4, c)\), where \( c = \min(b, 1) \), so it has range \((f(c), 4]\) on that interval, where \( f(c) < 4 \).

Since \((f(a), 4]\) \(\cap\) \((f(c), 4]\) = \((m, 4]\), where \( m = \max(f(a), f(c)) \) < 4, we see that the number \( \frac{m + 4}{2} < 4 \) and lies in the range of \( f \) on the interval \((a, -1]\) and on the interval \((-1, c) \subset (-1, b)\), so the horizontal line rule is not obeyed by \( f \) on the interval \((a, b)\) and \( f \) is not invertible on that interval.

- \( f \) is not invertible on any open interval containing the point \( x = 1 \), since on any interval of the form \((p, q)\) with \( p < 1 < q \), we have \( f \) increasing on the interval \([1, q]\), so it has range \([0, f(q)]\) on that interval, where \( f(q) > 0 \) and we have \( f \) decreasing on the interval \([r, 1]\), where \( r = \max(p, -1)\), so it has range \([0, f(r)]\) on that interval, where \( f(r) > 0 \).

Since \([0, f(q)] \cap [0, f(r)] = [0, s]\), where \( s = \min(f(q), f(r)) > 0 \), we see that the number \( \frac{s}{2} > 0 \) and lies in the range of \( f \) on the interval \((1, q)\) and on the interval \((r, 1) \subset (p, 1)\), so the horizontal line rule is not obeyed by \( f \) on the interval \((p, q)\) and \( f \) is not invertible on that interval.

- If \( U \) is any open interval, containing neither the point \(-1\), nor the point \( 1 \), then \( U \) is an open subset of one of the intervals \( A = (-\infty, -1) \), \( B = (-1, 1) \), or \( C = (1, \infty) \).

On each of these intervals \( f' \) is never zero and has a fixed sign (positive on \( A \) and on \( C \), negative on \( B \)), so on each of these intervals \( f \) is strictly monotonic, so has an inverse.

So \( f \) is invertible on an open interval \( U \) if and only if \( U \) contains neither the real \(-1\), nor the real \( 1 \) as an element.
**Question 7**

Let a function $f(x)$ be given by the formulas:

$$f(x) = x^2 + x^3 \cos \left( \frac{1}{x} \right), \text{ for any real } x \neq 0 \text{ and } f(0) = 0.$$  

Find all points where $f$ is differentiable and all points where the derivative of $f$ is continuous, with proof.

When $x \neq 0$, the function $f$ is clearly differentiable with continuous derivative, using the various derivative formulas:

$$f'(x) = 2x + 3x^2 \cos(x^{-1}) - x^3 \sin(x^{-1})(-x^{-2}) = 2x + 3x^2 \cos(x^{-1}) + x \sin(x^{-1}).$$

We note that $\lim_{x \to 0} f'(x) = 0$, since the terms $2x$, $3x^2$ and $x$ all go to zero as $x$ goes to zero, whereas the trigonometric terms are bounded.

Lastly, we compute the derivative of $f$ at the origin:

$$f'(0) = \lim_{x \to 0} \left( \frac{f(x) - f(0)}{x - 0} \right)$$

$$= \lim_{x \to 0} \left( \frac{f(x) - 0}{x} \right)$$

$$= \lim_{x \to 0} \left( \frac{f(x)}{x} \right)$$

$$= \lim_{x \to 0} \left( \frac{x^2 + x^3 \cos \left( \frac{1}{x} \right)}{x} \right)$$

$$= \lim_{x \to 0} \left( x + x^2 \cos \left( \frac{1}{x} \right) \right) = 0.$$  

Here we used that the terms $x$ and $x^2$ both go to zero, whereas the trigonometric term is bounded, as $x \to 0$.

Since we also have $\lim_{x \to 0} f'(x) = 0 = f'(0)$, we see that $f'$ is continuous at the origin, so $f$ is everywhere continuously differentiable and we are done.
Question 8

Suppose that $f$ is twice continuously differentiable and on the interval $[0, 2]$ we have $0 \leq f''(x) \leq 2$ and on the interval $[2, 4]$, we have $2 \leq f'(x) \leq 4$.
Given that $f(0) = 0$ and $f'(0) = -2$, what can we say about the possible values of $f(4)$? Explain your answer.

Apply the mean-value theorem to the interval $[0, x]$, where $x \leq 2$, for the function $f'(x)$, giving the formula:

$$f'(x) - f'(0) = (x - 0)f''(c), \quad c \in (0, x).$$

Since $0 \leq f''(c) \leq 2$, we get:

$$0 \leq f'(x) + 2 \leq 2x,$$
$$-2 \leq f'(x) \leq 2x - 2, \quad 0 \leq x \leq 2.$$

Putting $x = 2$, we get $-2 \leq f'(2) \leq 2$.
But we are given that $2 \leq f'(2) \leq 4$, so we conclude that $f'(2) = 2$.

Now apply the mean value theorem to the function $h(x) = f'(x) - 2x + 2$ on the interval $[0, x], x \leq 2$.
We have $h(0) = f'(0) + 2 = 0$ and $h'(x) = f''(x) - 2 \leq 0$.

$$h(x) - h(0) = x(f''(c) - 2), \quad c \in (0, x).$$

This gives $h(x) \leq 0$ for $x \in [0, 2]$.
But $h(0) = 0$ and $h(2) = f'(2) - 4 + 2 = 0$, so $h(0) = h(2) = 0$.
Also $h(x)$ is monotonic decreasing, since $h'(x) \leq 0$.

So $h(x)$ must be constant and identically zero.
So on the interval $[0, 2]$ we have $f'(x) = 2x - 2$.
Put $g(x) = f(x) - x^2 + 2x$.
Then $g(0) = f(0) = 0$ and $g'(x) = f'(x) - 2x + 2 = 0$, so $g(x)$ is constant, so is identically zero and we have on the interval $[0, 2], f(x) = x^2 - 2x$.
This gives $f(2) = 0$.
Also we have $f'(2) = 2$ and $f''(2) = 2$. 
We now consider \( f \) on the interval \([2, 4]\). We have \( f(2) = 0, f'(2) = 2, f''(2) = 2 \) and \( f \) is twice continuously differentiable.

We apply the mean value theorem to the interval \([2, 4]\), giving the relation:

\[
f(4) - f(2) = (4 - 2)f'(c), \quad c \in (2, 4),
\]

\[
f(4) = 2f'(c), \quad c \in (2, 4).
\]

Since we know that \( 2 \leq f'(c) \leq 4 \) we conclude that: \( 4 \leq f(4) \leq 8 \).

Next consider the function, with domain \([2, 4]\):

\[
p(x) = f(x) - 4x + 8, \quad x \in [2, 4].
\]

We have \( p(2) = f(2) - 8 + 8 = 0 \) and \( p'(2) = f'(2) - 4 = 2 - 4 = -2 \).

Also on the interval \([2, 4]\), we have \( p'(x) = f'(x) - 4 \), so, since \( 2 \leq f'(x) \leq 4 \), we have \(-2 \leq p'(x) \leq 0\).

In particular \( p \) is monotonic decreasing on \([2, 4]\).

Since \( p(2) = 0 \) and \( p'(2) < 0 \), we have \( p(x) < 0 \) on some interval \([2, c]\), for some real \( c \) with \( 2 < c \leq 4 \).

In particular \( p(c) < 0 \) and then, since \( p \) is monotonic decreasing on \([c, 4]\), we have \( p(4) \leq p(c) < 0 \).

But \( p(4) = f(4) - 16 + 8 = f(4) - 8 \).

So \( f(4) - 8 < 0 \). So \( f(4) < 8 \).

So we have proved that \( f(4) \) must lie in the range: \( 4 \leq f(4) < 8 \).

Next consider the function \( q(x) = f(x) - 2x + 4 \), defined for \( 2 \leq x \leq 4 \).

We have \( q(2) = f(2) - 4 + 4 = 0, q'(2) = f'(2) - 2 = 0 \) and \( q''(2) = f''(2) = 2 \).

Also we have \( q'(x) = f'(x) - 2 \geq 0 \).

So \( q \) is monotonic increasing.

Since \( q'(2) = 0 \) and \( q''(2) > 0 \), we see that on some interval \([2, d]\), with \( 2 < d \leq 4 \), we have \( q'(x) > 0 \).

It follows that \( q \) is strictly increasing on the interval \([2, d]\).

Since \( q(2) = 0 \), we have \( q(d) > q(2) \), so \( q(d) > 0 \).

Since \( q \) is monotonic increasing, we have \( q(4) \geq q(d) > 0 \), so \( q(4) > 0 \).

But \( q(4) = f(4) - 8 + 4 = f(4) - 4 \).

So \( f(4) - 4 > 0 \), so \( f(4) > 4 \).

So we have proved that \( f(4) \) must lie in the range: \( 4 < f(4) < 8 \).

It is fairly easy to convince oneself that all values of \( f(4) \) in the interval \((4, 8)\) are attainable by suitable functions \( f \).