Theoretical Mathematics II, Exam 1, 10/15/9

Name:
Show your work.
Twenty points per question.
The best five questions will count.

Question 1

Let \( f(x) = \frac{1}{\sqrt{x}} \).

Prove from first principles the following limit:

\[
\lim_{x \to 9} f(x) = \frac{1}{3}.
\]

Also find an open interval \( K \), containing the number 9, such that, for any \( x \in K \), \( f(x) \) is within \( \frac{1}{6} \) of \( \frac{1}{3} \).
Question 2

Prove from first principles the following limit:

$$\lim_{x \to 1} \left( \frac{2x - 3}{x - 2} \right) = 1.$$ 

Use your result to determine the following limit, explaining your reasoning:

$$\lim_{x \to 1} \sin \left( \frac{2x - 3}{x - 2} \right).$$
Question 3

Determine, with proof, each of the following limits, or prove that the limit in question does not exist:

- \( A = \lim_{x \to 5} \left( \frac{x^2 - 3x - 10}{25 - x^2} \right) \)

- \( B = \lim_{x \to 0} \left( x \sin^2 \left( \frac{1}{x} \right) + x^2 \cos(x) \right) \)
Question 4

Let $h : \mathbb{R} \to \mathbb{R}$ be such that $h(x) = 2x + 3$ if $x$ is rational and $h(x) = x^2$, if $x$ is irrational.

Describe the range of the function $h$.

Also find, with proof all points, if any, where the function $h(x)$ is continuous.
Question 5

Using the limits given on page 109 of the text and basic properties of trigonometric functions, prove the limits:

$$\lim_{x \to 0} \left( \frac{1 - \cos(x)}{x \sin(x)} \right) = \frac{1}{2}.$$

$$\lim_{x \to 0} \left( \frac{\sin(2x)}{\sqrt{1 + x} - \sqrt{1 - x}} \right) = 2.$$

Also prove that the following limit does not exist:

$$\lim_{x \to \infty} x \sin(x^2).$$
Question 6

Suppose that \( \lim_{x \to \infty} f(x) = 4 \).

Prove from first principles that \( \lim_{x \to \infty} \frac{1}{f(x)} = \frac{1}{4} \).

Now suppose instead that \( \lim_{x \to \infty} f(x) = 0 \).

Does it follow that \( \lim_{x \to \infty} \frac{1}{f(x)} = \infty \)?

Explain your answer, with an example.
Question 7

Let \( f(x) = x^3 + 4x^2 - 5 \).

Prove that the function \( f \) is increasing on the interval \([1, 3]\).

Hence determine, with proof, the range of the function \( f \) for the domain \([1, 3]\).

Also prove that for the domain \([-3, 3]\), the function \( f \) has no inverse.

(Hint for this part: look at the signs of \( f \) at integer points in this range).
Question 8
Let $f(x) = x^2 - 5$.
Prove that $f$ has a root in the interval $J = [2, 3]$.
By the method of repeated bisection of intervals, beginning with the interval $J$, determine the location of the positive root of the equation $f(x) = 0$, with an accuracy of at least 0.0625.